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Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian

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Abstract

Let G be a simple algebraic group defined over $\mathbb{C}$ and T be a maximal torus of G. For a dominant coweight λ of G, the T-fixed point subscheme $(\overline{\mathrm{Gr}}_G^\lambda)^T$ of the Schubert variety $\overline{\mathrm{Gr}}_G^\lambda$ in the affine Grassmannian Gr_G is a finite scheme. We prove that for all such λ if G is of type A or D and for many of them if G is of type E, there is a natural isomorphism between the dual of the level one affine Demazure module corresponding to λ and the ring of functions (twisted by certain line bundle on Gr_G) of $(\overline{\mathrm{Gr}}_G^\lambda)^T$. We use this fact to give a geometrical proof of the Frenkel–Kac–Segal isomorphism between basic representations of affine algebras of A, D, E type and lattice vertex algebras.
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Keywords: Basic representation; Frenkel–Kac–Segal isomorphism; Affine Grassmannian

0. Introduction

0.1. The Frenkel–Kac–Segal isomorphism

In their fundamental papers [14,30], Frenkel–Kac and Segal constructed the bosonic realizations of the basic representations of simply-laced affine algebras, using the vertex operators of string theory. Let us first review their theorem.

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0.1.1. Let $\mathfrak{g}$ be the Lie algebra of a simple algebraic group G over $\mathbb{C}$, and $(\,,)$ denote the normalized invariant form on $\mathfrak{g}$. The untwisted affine Kac–Moody algebra associated to $(\mathfrak{g}, (\,,))$ is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}K$$

where K is central in $\hat{\mathfrak{g}}$ and

$$\left[X \otimes t^m, Y \otimes t^n\right] = [X, Y] \otimes t^{m+n} + m\delta_{m,-n}(X, Y)K.$$

Recall that the level k vacuum module of $\hat{\mathfrak{g}}$ is defined as

$$\mathbb{V}(k\Lambda) = \mathrm{Ind}_{\mathfrak{g}\otimes\mathbb{C}[t]+\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}$$

on which $\mathfrak{g} \otimes \mathbb{C}[t]$ acts through the trivial character and K acts by multiplication by k. $\mathbb{V}(k\Lambda)$ has a vertex algebra structure. When k is a positive integer, the vacuum module has a unique irreducible quotient $L(k\Lambda)$, which is an integrable $\hat{\mathfrak{g}}$-module of level k. Furthermore, it is a quotient vertex algebra of $\mathbb{V}(k\Lambda)$.

On the other hand, choose $\mathfrak{t} \subset \mathfrak{g}$ to be a Cartan subalgebra of $\mathfrak{g}$. Let $\iota : \mathfrak{t} \to \mathfrak{t}^*$ be the isomorphism induced by the bilinear form on $\mathfrak{t}$ induced from the normalized invariant form on $\mathfrak{g}$. The restriction of the central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ to $\mathfrak{t} \otimes \mathbb{C}[t, t^{-1}]$ defines the Heisenberg Lie algebra $\hat{\mathfrak{t}} \subset \hat{\mathfrak{g}}$. Let $R_G \subset \mathfrak{t}$ be the coroot lattice of G. Define a module over $\hat{\mathfrak{t}}$ by

$$V_{R_G} = \bigoplus_{\lambda\in R_G} \pi_\lambda$$

where π_λ is the level one Fock module of $\hat{\mathfrak{t}}$ with highest weight $\iota\lambda$ (see 3.1.1). When G is simply-laced, there is a unique (up to isomorphism) simple vertex algebra structure on V_{R_G}, corresponding to the cohomology class $H^2(R_G, \mathbb{C}^*)$ determined by the invariant form.

Now the result of Frenkel–Kac and Segal may be stated as follows (in the language of vertex algebras).

0.1.2. Theorem. *If $\mathfrak{g}$ is a simple Lie algebra of type A, D or E, then $L(\Lambda) \cong V_{R_G}$ as vertex algebras.*

In particular, we could obtain a character formula for $L(\Lambda)$.

0.1.3. Corollary. *$L(\Lambda)$ is isomorphic to V_{R_G} as $\hat{\mathfrak{t}}$-modules.*

We should point out that we reverse the historical order here. In fact, this corollary was obtained before the FKS isomorphism (cf. [19, (3.37)]), and was used to prove the FKS isomorphism.

0.2. Main results

The goal of the present paper is to interpret the FKS isomorphism from an algebro-geometrical point of view.

0.2.1. The starting point is the Borel–Weil theorem for affine Kac–Moody algebras, which was originally proved in [24] and [27]. For any algebraic group G over $\mathbb{C}$, let $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ be the affine Grassmannian of G, where $G_{\mathcal{K}}$ is group of maps from the punctured disc to G and $G_{\mathcal{O}}$ is the group of maps from the disc to G. When G is simple, simply-connected, there is an ample invertible sheaf $\mathcal{L}_G$ on Gr_G, which is a generator of the Picard group of Gr_G. The Borel–Weil type theorem identifies $L(k\Lambda)$ with $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k})^*$. On the other hand, for a chosen maximal torus $T \subset G$, the affine Grassmannian Gr_T of T naturally embeds into Gr_G, and V_{R_G} is then identified with $\Gamma(\mathrm{Gr}_G, \mathcal{O}_{\mathrm{Gr}_T} \otimes \mathcal{L}_G)^*$. Therefore, the question now is whether the natural morphism

$$\mathcal{L}_G \to \mathcal{O}_{\mathrm{Gr}_T} \otimes \mathcal{L}_G \tag{1}$$

induces an isomorphism between the spaces of their global sections, if G is simply-laced.

It turns out that we can push the question further. Recall that Gr_G is stratified by $G_{\mathcal{O}}$-orbits, which are parameterized by dominant coweights. For a dominant coweight λ, the corresponding $G_{\mathcal{O}}$-orbit is denoted by Gr_G^λ. Let $\overline{\mathrm{Gr}}_G^\lambda$ be the closure of Gr_G^λ in Gr_G. Since G is simple, $\mathrm{Gr}_G = \varinjlim \overline{\mathrm{Gr}}_G^\lambda$. Then one could ask, whether the restriction of (1) to each $\overline{\mathrm{Gr}}_G^\lambda$ will still induce an isomorphism between the spaces of their global sections. Let us reformulate the question slightly differently. The maximal torus T of G acts on the affine Grassmannian Gr_G as well as on each Schubert variety $\overline{\mathrm{Gr}}_G^\lambda$. It will be shown that the embedding $\mathrm{Gr}_T \subset \mathrm{Gr}_G$ identifies Gr_T as the T-fixed point subscheme of Gr_G. Therefore, $\mathrm{Gr}_T \times_{\mathrm{Gr}_G} \overline{\mathrm{Gr}}_G^\lambda$ is the T-fixed point subscheme $(\overline{\mathrm{Gr}}_G^\lambda)^T$ of $\overline{\mathrm{Gr}}_G^\lambda$. Then the restriction of (1) to $\overline{\mathrm{Gr}}_G^\lambda$ becomes

$$\mathcal{O}_{\overline{\mathrm{Gr}}_G^\lambda} \otimes \mathcal{L}_G \to \mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T} \otimes \mathcal{L}_G. \tag{2}$$

The main theorem of this paper is

0.2.2. Theorem. *Let G be a simple, connected (not necessarily simply-connected) algebraic group. Let Gr_G be the affine Grassmannian of G and $\overline{\mathrm{Gr}}_G^\lambda$ be the Schubert variety associated to a dominant coweight λ. Fix $T \subset G$ a maximal torus, and let $(\overline{\mathrm{Gr}}_G^\lambda)^T$ be the T-fixed subscheme of $\overline{\mathrm{Gr}}_G^\lambda$. Let $\mathcal{L}_G$ be the ample invertible sheaf on Gr_G, which is the generator of the Picard group of each connected component. Then for all λ if G is of type A or D, and for many of then if G is of type E, the natural morphism* (2) *induces an isomorphism*

$$\Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G\big) = \Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T} \otimes \mathcal{L}_G\big).$$

We expect that the theorem still holds for all λ if G is of type E, for the FKS isomorphism holds for all simply-laced algebraic groups. However, so far only partial results are obtained for this type, see 2.2.16, 2.2.17. Observe that the theorem could not hold for non-simply-laced algebraic groups.

0.2.3. One of the applications of the above theorem is to the study of the singularities of Schubert varieties in the affine Grassmannian. In principle, for any algebraic variety with a torus action, the singularity at a fixed point under the action is reflected by the local ring of the fixed point subscheme at that point. In particular, we will prove that

0.2.4. Corollary. *Let G be a simple algebraic group of type A or D. Then the smooth locus of $\overline{\mathrm{Gr}}_G^\lambda$ is Gr_G^λ.*

In fact, this holds for any simple algebraic group, as is shown in [10] and [26]. However, since we will deduce it as a corollary of our main theorem, we will confine ourselves to algebraic groups of type A or D. Remark that the same statement of our main theorem for algebraic groups of type E will imply the above corollary for them as well.

0.2.5. The main application of Theorem 0.2.2 is an alternative proof of the FKS isomorphism, which is applicable to all simply-laced algebraic groups. More precisely, we will prove a geometrical version of the isomorphism using Beilinson and Drinfeld's idea of factorization algebras (cf. [5]). In addition, the main theorem also allows us to identify the modules over the basic representations with the modules over the lattice vertex algebras.

Assume that G is of adjoint type. Let Λ_G be the coweight lattice of G. It is known (cf. [9]) that the simple modules over V_{R_G} are labelled by $\gamma \in \Lambda_G/R_G$. Let $V_{R_G}^\gamma$ denote the one corresponding to $\gamma \in \Lambda_G/R_G$. As $\hat{\mathfrak{t}}$-modules,

$$V_{R_G}^\gamma = \bigoplus_{\lambda\in\gamma+R_G} \pi_\lambda.$$

It is known that for any $\gamma \in \Lambda_G/R_G$, there is a unique minuscule fundamental coweight ω_{i_γ}, which is a representative of the coset γ. Then the simple $\hat{\mathfrak{g}}$-module $L(\Lambda + \iota\omega_{i_\gamma})$ of highest weight $\Lambda + \iota\omega_{i_\gamma}$ has a module structure over $L(\Lambda)$. We have

0.2.6. Theorem. *Under the identification of $L(\Lambda) \cong V_{R_G}$ of vertex algebras, we have an isomorphism*

$$L(\Lambda + \iota\omega_{i_\gamma}) \cong V_{R_G}^\gamma$$

as modules over them.

0.3. New perspectives of the FKS isomorphism

Our geometrical interpretation of the FKS isomorphism brings us more insight into this fundamental theorem. We briefly indicate some of the new perspectives in this introduction. Details and further discussions will appear in [31]. The rest of the paper is independent of this subsection.

0.3.1. As we point out, the FKS isomorphism (more precisely, Corollary 0.1.3) amounts to

$$\Gamma(\mathrm{Gr}_G, \mathcal{L}_G) \cong \Gamma(\mathrm{Gr}_T, \mathcal{L}_G|_{\mathrm{Gr}_T}).$$

The crucial observation is that instead of interpreting Gr_T as the T-fixed point subcheme of Gr_G, we can also interpret it as the affine Springer fiber (cf. [23] and [16]) in Gr_G corresponding to the regular semisimple element ρ in $\mathfrak{g}(\mathcal{K})$. This naturally leads to the following consideration. Let $u \in \mathfrak{g}(\mathcal{K})$ be any regular semi-simple element and let $\mathrm{Gr}_{G,u}$ be the corresponding affine Springer fiber. We can ask the similar question whether the natural map

$$\Gamma(\mathrm{Gr}_G, \mathcal{L}_G) \to \Gamma(\mathrm{Gr}_{G,u}, \mathcal{L}_G|_{\mathrm{Gr}_{G,u}}) \tag{3}$$

is an isomorphism in the case when G is simply-laced. Furthermore, whether the restriction of the above map to each Schubert variety $\overline{\mathrm{Gr}}_G^{\lambda}$

$$\Gamma\big(\overline{\mathrm{Gr}}_G^{\lambda}, \mathcal{L}_G\big) \to \Gamma\big(\overline{\mathrm{Gr}}_G^{\lambda} \cap \mathrm{Gr}_{G,u}, \mathcal{L}_G|_{\overline{\mathrm{Gr}}_G^{\lambda} \cap \mathrm{Gr}_{G,u}}\big) \tag{4}$$

is an isomorphism. In the following, we will see why these are interesting questions to ask.

0.3.2. Let J_u be the centralizer of u in $G_{\mathcal{K}}$. This is a maximal torus of $G_{\mathcal{K}}$, which is usually non-split. It is well known (cf. [23]) that the conjugacy classes maximal tori in $G_{\mathcal{K}}$ are parameterized by conjugacy classes of the Weyl group of G. For example, if $u = \rho$, then J_ρ is a split torus which corresponds to the identity element in the Weyl group while if $u = p = \sum e_{\check{\alpha}_i} + t f_{\check{\theta}}$, J_p corresponds to the Coxeter element.

The $\mathbb{G}_m$-central extension of $G_{\mathcal{K}}$ gives rise to a $\mathbb{G}_m$-central extension of J_u, called the Heisenberg group, and denoted by $\hat{J}_u$. For example, $\hat{J}_\rho$ is called the homogeneous Heisenberg group and $\hat{J}_p$ is called the principal Heisenberg group. The dual statement of (3) says that $L(\Lambda)$ is isomorphic to $\Gamma(\mathrm{Gr}_{G,u}, \mathcal{L}_G|_{\mathrm{Gr}_{G,u}})^*$ as $\hat{J}_u$-modules. If $\mathrm{Gr}_{G,u}$ is 0-dimensional, one can show that $\Gamma(\mathrm{Gr}_{G,u}, \mathcal{L}_G|_{\mathrm{Gr}_{G,u}})^*$ is an irreducible $\hat{J}_u$ module. Therefore, (3) implies that $L(\Lambda)$ remains irreducible as a $\hat{J}_u$-module, for any type of Heisenberg group $\hat{J}_u$. In particular, if $u = \rho$, this is the FKS isomorphism and if $u = p$, this is the so-called principal realization of the basic representation, which was proved in [20]. (We remark here that there is also a statement for $u = p$ similarly to Theorem 0.2.2, which can be obtained by the method similarly to method of this paper.) However, the types of such Heisenberg groups are parameterized by conjugacy classes of the Weyl group of G. We thus obtain for any conjugacy class of the Weyl group, a realization of the basic representation. Indeed, this has already been considered in [21].

0.3.3. Now we give another interesting indication of (3). For this, we should first recall the finite-dimensional story. Let $\check{\lambda}$ be a dominant weight of G. We denote $P_{\check{\lambda}}$ be the parabolic subgroup of G such that the stabilizer $W_{\check{\lambda}}$ of $\check{\lambda}$ in the Weyl group W of G is the same as the Weyl group $W_{P_{\check{\lambda}}}$ of $P_{\check{\lambda}}$. Let $\mathcal{P}_{\check{\lambda}}$ be the variety of parabolic subgroups of G of type $P_{\check{\lambda}}$. Let $\mathcal{O}(\check{\lambda})$ be the invertible sheaf on $\mathcal{P}_{\check{\lambda}}$ such that $\Gamma(\mathcal{P}_{\check{\lambda}}, \mathcal{O}(\check{\lambda}))^*$ the irreducible G-module of highest weight $\check{\lambda}$. Let $\xi \in \mathfrak{g}$ be any regular element, and $\mathcal{P}_{\check{\lambda}}^{\xi}$ be the corresponding Springer fiber (which is a finite subscheme of $\mathcal{P}_{\check{\lambda}}$). One can prove that the natural map

$$\Gamma\big(\mathcal{P}_{\check{\lambda}}, \mathcal{O}(\check{\lambda})\big) \to \Gamma\big(\mathcal{P}_{\check{\lambda}}^{\xi}, \mathcal{O}(\check{\lambda})|_{\mathcal{P}_{\check{\lambda}}^{\xi}}\big) \tag{5}$$

is always surjective. It is an isomorphism if and only if $\check{\lambda}$ is minuscule weight of G. It is clear that (3) is the affine counterpart of the above statement. Therefore, it suggests to us to call Λ (as well as $\Lambda + \omega_{i_\gamma}$) a minuscule weight of $\hat{\mathfrak{g}}$ when $\mathfrak{g}$ is simply-laced. Observe that in non-simply-laced case, it seems no weights of $\hat{\mathfrak{g}}$ should be called minuscule. (One should consider the twisted affine algebras.)

Let us also point out that in finite-dimensional story, the isomorphism (5) for $\check{\lambda}$ minuscule also reflects the fact that the Schubert variety $\overline{\mathrm{Gr}}_{{}^LG}^{\check{\lambda}}$ for the Langlands dual group ${}^LG$ is smooth.

From (3), one also expects that the Schubert variety corresponding to Λ in the double affine Grassmannian of the Langlands dual group of $\hat{G}_{\mathcal{K}}$ should also be smooth in some sense.

0.4. Contents

The paper is organized as follows.

In Section 1, we collect various facts related to affine Grassmannians that are needed in the sequel. In Section 1.1, we recall some geometry of affine Grassmannians. In Section 1.2, we construct a flat degeneration of $\overline{\mathrm{Gr}}^{\lambda}_{G} \times \overline{\mathrm{Gr}}^{\mu}_{G}$ to $\overline{\mathrm{Gr}}^{\lambda+\mu}_{G}$. While it is not difficult to produce such a flat family, it is not trivial to prove that the special fiber is reduced. It is based on the tensor structure of affine Demazure modules, first discovered in [12] (see Theorem 1 of [12]) by combinatoric methods. We reprove their result here in a purely algebro-geometrical way (Theorem 1.2.2). In Section 1.3, we show that the embedding $\mathrm{Gr}_T \to \mathrm{Gr}_G$ identifies Gr_T as the T-fixed point subscheme of Gr_G. In Section 1.4, we prove a Borel–Weil type theorem for the nonneutral components of the affine Grassmannians.

Section 2 is devoted to the proof of our main theorem, Theorem 0.2.2. In Section 2.1, we reduce the full theorem to the special cases where λ is a fundamental coweight, so that the geometry of the corresponding Schubert variety is relatively simple. In Section 2.2, we prove the theorem for fundamental coweights of algebraic groups of type A and D. We will also have a brief discussion on some partial results for the algebraic groups of type E. In Section 2.3, we prove 0.2.4 as a simple application of our main theorem.

In Section 3, we return to our motivation of the paper. In Section 3.1, we deduce from the main theorem that $L(\Lambda + \iota\omega_{i_\gamma})$ is isomorphic to $V^{\gamma}_{R_G}$ as $\hat{\mathfrak{t}}$-modules if G is of type A, D, E. Then we recall some basic ingredients of lattice factorization algebras and affine Kac–Moody factorization algebras in Section 3.2. Finally, we reprove the FKS isomorphism as well as Theorem 0.2.6 in Section 3.3.

0.5. Notation and conventions

Throughout this paper, we will work over the base field $\mathbb{C}$. However, all results remain true over any algebraically closed field of characteristic 0. A $\mathbb{C}$-algebra will always be assumed to be associative, commutative and unital.

If X is a scheme (or a space) over some base S, and $S' \to S$ is a morphism, we denote $X_{S'}(\text{or } X|_{S'}) = X \times_S S'$ the base change of X to S'.

If G is a group scheme, $\mathcal{F}$ is a G-torsor over some base, and X is a G-scheme, we write $\mathcal{F} \times^G X$ for the associated product.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[\![t]\!]$. If G is an algebraic group, we will denote by $G_{\mathcal{O}}$ the group scheme whose $\mathbb{C}$-points are $G(\mathcal{O})$ and by $G_{\mathcal{K}}$ the ind-group whose $\mathbb{C}$-points are $G(\mathcal{K})$. The neutral connected component of $G_{\mathcal{K}}$ is denoted by $G^0_{\mathcal{K}}$.

If G is a reductive group, we will denote Λ_G (respectively R_G, respectively Λ^+_G) the coweight lattice (respectively the coroot lattice, respectively dominant coweights) of G and $\check{?}$ the dually named object corresponding to ?. If $\check{\nu} \in \check{\Lambda}^+_G$ is a dominant weight, we denote $V^{\check{\nu}}$ the irreducible representation of G of highest weight $\check{\nu}$. However, if G is a torus, then $V^{\check{\nu}}$ is 1-dimensional, and will be denoted as $\mathbb{C}^{\check{\nu}}$. The Weyl group of G is denoted by W.

If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$, we will denote by I the set of vertices of the Dynkin diagram of $\mathfrak{g}$. Then fundamental coweight (respectively fundamental weight, respectively simple coroot, respectively simple root) of $\mathfrak{g}$ corresponding to $i \in I$ is denoted (or rather, chosen) to be ω_i (respectively $\check{\omega}_i$, respectively α_i, respectively $\check{\alpha}_i$). The highest root of $\mathfrak{g}$ is denoted by $\check{\theta}$ and the corresponding coroot by θ. Remark that θ is usually not the highest coroot. The invariant bilinear form $(\,,\,)$ on $\mathfrak{g}$ is normalized so that the corresponding bilinear form $(\,,\,)^*$ on the dual of Cartan subalgebra of $\mathfrak{g}$ satisfies that the square of the length of the long root is 2. Then the invariant form induces an isomorphism from the Cartan subalgebra to its dual, which is denoted by ι. The untwisted affine Kac–Moody algebra associated to $(\mathfrak{g}, (\,,\,))$ is denoted by $\hat{\mathfrak{g}}$. Denote by i_0 the extra vertex in the affine Dynkin diagram. The fundamental weight of $\hat{\mathfrak{g}}$ corresponding to i_0 is denoted by Λ. Denote by K the central element in $\hat{\mathfrak{g}}$ such that $\Lambda(K) = 1$. Denote by $\check{\delta}$ the imaginary root of $\hat{\mathfrak{g}}$ such that $\check{\alpha}_0 = \check{\delta} - \check{\theta}$ is the simple root corresponding to i_0.

1. Affine Grassmannians

We recall the definition and the basic properties of the affine Grassmannian associated to an algebraic group in Section 1.1. Then we study a flat degeneration of Schubert varieties in Section 1.2, the fixed point subscheme of the affine Grassmannian in Section 1.3 and the Borel–Weil theorem in Section 1.4.

1.1. Affine Grassmannians

1.1.1. The affine Grassmannian Gr_G of *any* affine algebraic group G is defined to be the *fppf* quotient $G_{\mathcal{K}}/G_{\mathcal{O}}$. Without loss of generality, we could and will assume that G is connected in the rest of the paper. We collect here some facts most relevant to our application in this paper, and refer to [6, Section 4.5], [3] and [25] for a general discussion of affine Grassmannians.

1.1.2. Theorem.

(1) Gr_G *is an ind-scheme of ind-finite type, and ind-projective if G is reductive.*
(2) Gr_G *is reduced if and only if* $\mathrm{Hom}(G, \mathbb{G}_m) = 0$.
(3) *There is a bijection* $\pi_0(\mathrm{Gr}_G) = \pi_1(G)$, *and different connected components of* Gr_G *are isomorphic as ind-schemes.*
(4) *If G is simple and simply-connected, then Pic*$(\mathrm{Gr}_G) = \mathbb{Z}$.

1.1.3. Remark. In the paper, we will consider the scheme structure of Gr_G. By (2) of above theorem, if G is a connected semisimple algebraic group, Gr_G is reduced. On the other hand, if $G = T$ is a torus, then $(\mathrm{Gr}_T)_{\mathrm{red}} = X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ is a discrete space, but Gr_T itself is nonreduced. Indeed, the connected component of $1 \in T(\mathcal{K})/T(\mathcal{O})$ is the formal group with Lie algebra $\mathfrak{t}(\mathcal{K})/\mathfrak{t}(\mathcal{O})$, where $\mathfrak{t}$ is the Lie algebra of T.

1.1.4. Assume that G is reductive. We choose $T \subset G$, a maximal torus. Recall that Gr_G is stratified by the $G_{\mathcal{O}}$-orbits, labelled by Λ_G^+. For any $\lambda \in \Lambda_G = \mathrm{Hom}(\mathbb{G}_m, T)$, a choice of a uniformizer $t \in \mathcal{O}$ determines a closed point $t^\lambda \in G_{\mathcal{K}}$. Let s^λ be the image of t^λ under the map $G_{\mathcal{K}} \to \mathrm{Gr}_G$. Then s^λ is a well-defined point in Gr_G, which does not depend on the choice of the uniformizer t. Let $\mathrm{Gr}_G^\lambda = G_{\mathcal{O}} s^\lambda$. Remark that Gr_G^λ does not depend on the choice of the maximal torus. Then $\mathrm{Gr}_G(\mathbb{C})$ is the union of $\mathrm{Gr}_G^\lambda(\mathbb{C})$ for $\lambda \in \Lambda_G^+$. Let $\overline{\mathrm{Gr}}_G^\lambda$ its closure in Gr_G. It is known that $\overline{\mathrm{Gr}}_G^\mu \subset \overline{\mathrm{Gr}}_G^\lambda$ if and only if $\mu \leqslant \lambda$. If Gr_G is reduced, e.g. G is semisimple, then as ind-schemes, $\mathrm{Gr}_G \cong \varinjlim \overline{\mathrm{Gr}}_G^\lambda$, where the limit is taken over $\lambda \in \Lambda_G^+$.

1.1.5. Let G be a reductive group. As in [2], there is a moduli interpretation of the affine Grassmannian Gr_G. Fix a smooth curve X, and a closed point $x \in X$. Denote $X^* = X - \{x\}$. Then

$$\mathrm{Gr}_{G,x}(R) = \{\mathcal{F} \text{ a } G\text{-torsor on } X_R, \beta : \mathcal{F}_{X_R^*} \to X_R^* \times G \text{ a trivialization on } X_R^*\}$$

is represented by an ind-scheme which is isomorphic to Gr_G, once an isomorphism $\mathcal{O}_x \cong \mathcal{O}$ is chosen, where $\mathcal{O}_x$ is the complete local ring of x. Remark that if $T \subset G$ a maximal torus is chosen, s^λ is represented by the pair $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a T-torsor over X, and β is as above, such that for any $\check{\nu} \in \check{\Lambda}_G$, the induced invertible sheaf and trivialization are

$$\beta_\nu : \mathcal{F} \times^T \mathbb{C}^{\check{\nu}} \cong \mathcal{O}_X(\langle\lambda, \check{\nu}\rangle x)$$

we could also attach for any closed point $x \in X$, the group scheme $G_{\mathcal{O},x} \cong G_{\mathcal{O}}$ as the group of trivializations of the trivial G-torsor on the disc $\operatorname{Spec} \mathcal{O}_x$, and the corresponding $G_{\mathcal{O},x}$-invariant subvarieties $\mathrm{Gr}^\lambda_{G,x} \subset \overline{\mathrm{Gr}}^\lambda_{G,x} \subset \mathrm{Gr}_{G,x}$.

1.1.6. If we allow the point x to vary in previous construction, we obtain a global version of the affine Grassmannian. Indeed, we can construct much more sophisticated geometrical objects thanks to the moduli interpretation of the affine Grassmannian. This is the so-called Beilinson–Drinfeld Grassmannian (cf. [6,29]). Denote the n-fold product of X by $X^n = X \times \cdots \times X$, and consider the functor

$$\mathrm{Gr}_{G,X^n}(R) = \left\{\begin{array}{l} (x_1, \ldots, x_n) \in X^n(R),\ \mathcal{F} \text{ a } G\text{-torsor on } X_R, \\ \beta_{(x_1,\ldots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } X_R - \bigcup_i x_i \end{array}\right\}.$$

Here we think of the points $x_i : \operatorname{Spec} R \to X$ as subschemes of X_R by taking their graphs. This functor is represented by an ind-scheme which is formally smooth over X^n. The one relevant to our case is Gr_{G,X^2}. Then, for a closed point $(x, y) \in X^2$,

$$(\mathrm{Gr}_{G,X^2})_{(x,y)} \cong \begin{cases} \mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,y} & \text{if } x \neq y, \\ \mathrm{Gr}_{G,x} & \text{if } x = y. \end{cases}$$

There is also a global version of $G_{\mathcal{O}}$, which is

$$G_{\mathcal{O},X^n}(R) = \left\{\begin{array}{l} (x_1, \ldots, x_n) \in X^n(R),\ \gamma_{(x_1,\ldots,x_n)} \text{ a trivialization} \\ \text{of the trivial } G\text{-torsor } \mathcal{F}_0 \text{ on } \widehat{\bigcup_i x_i} \end{array}\right\}$$

where $\widehat{\bigcup_i x_i}$ is the formal completion of X_R along $x_1 \cup \cdots \cup x_n$. $G_{\mathcal{O},X^n}$ is represented by a formally smooth group scheme over X^n. Like Gr_{G,X^2}, if $x \neq y$, $(G_{\mathcal{O},X^2})_{(x,y)} \cong G_{\mathcal{O},x} \times G_{\mathcal{O},y}$, and $(G_{\mathcal{O},X^2})_{(x,x)} \cong G_{\mathcal{O},x}$. Furthermore, $G_{\mathcal{O},X^n}$ acts on Gr_{G,X^n} in the following way. Let $(\mathcal{F}, \beta) \in \mathrm{Gr}_{G,X^n}$ and $\gamma \in G_{\mathcal{O},X^2}$, we construct $(\mathcal{F}', \beta) = \gamma \cdot (\mathcal{F}, \beta)$ as follows. If $\mathcal{F}$ can be trivialized along $\widehat{\bigcup_i x_i}$, then choose a trivialization and one obtains a transition function on $\widehat{\bigcup_i x_i} - \bigcup_i x_i$. Modifying this transition function by γ and gluing $\mathcal{F}|_{X_R - \bigcup x_i}$ and $\mathcal{F}_0|_{\widehat{\bigcup_i x_i}}$ by this new transition function, one obtains $(\mathcal{F}', \beta)$. If $\mathcal{F}$ cannot be trivialized along $\widehat{\bigcup_i x_i}$, choose some faithfully flat $R \to R'$, apply the same procedure and then use descent.

1.1.7. The $G_{\mathcal{O},X}$-orbits on $\mathrm{Gr}_{G,X}$ are parameterized by Λ_G^+. This is just a direct global counterpart of 1.1.4. We will denote the orbit corresponding to $\lambda \in \Lambda_G^+$ by $\mathrm{Gr}_{G,X}^\lambda$ and its closure by $\overline{\mathrm{Gr}}_{G,X}^\lambda$. Observe that there is a natural action of $G_{\mathcal{O},X}$ on $\overline{\mathrm{Gr}}_{G,X}^\lambda$.

1.1.8. Next, we turn to the $G_{\mathcal{O},X^2}$-orbits on Gr_{G,X^2}. They are parameterized by $(\lambda,\mu) \in \Lambda_G^+ \times \Lambda_G^+$. Choose a maximal torus T of G. For any $(\lambda,\mu) \in \Lambda_G \times \Lambda_G$, let $s^{\lambda,\mu} \in \mathrm{Gr}_{G,X^2}(X^2)$ be $(\mathrm{pr}_1, \mathrm{pr}_2, \mathcal{F}, \beta)$, where $\mathrm{pr}_1, \mathrm{pr}_2$ are two projections of X^2 to X, $(\mathcal{F},\beta)$ the T-torsor over X^3 with the trivialization β over $X^3 - \Delta_{13} \cup \Delta_{23}$, $\Delta_{13} = \{(x,y,x) \in X^3\}$ (respectively $\Delta_{23} = \{(y,x,x) \in X^3\}$) being the graph of pr_1 (respectively pr_2), such that for any $\check{\nu} \in \check{\Lambda}_G$, the induced invertible sheaves are

$$\beta_\nu : \mathcal{F} \times^T \mathbb{C}^{\check{\nu}} \cong \mathcal{O}_{X^3}\big(\langle\lambda,\check{\nu}\rangle\Delta_{13} + \langle\mu,\check{\nu}\rangle\Delta_{23}\big).$$

Then $s^{\lambda,\mu}$ is a section of $\mathrm{Gr}_{G,X^2} \to X^2$. Denote $\mathrm{Gr}_{G,X^2}^{\lambda,\mu} = G_{X^2,\mathcal{O}} s^{\lambda,\mu}$. Then $\mathrm{Gr}_{G,X^2}(\mathbb{C})$ is the union of $\mathrm{Gr}_{G,X^2}^{\lambda,\mu}(\mathbb{C})$ for $(\lambda,\mu) \in \Lambda_G^+ \times \Lambda_G^+$. Let $\overline{\mathrm{Gr}}_{G,X^2}^{\lambda,\mu}$ be the closure of $\mathrm{Gr}_{G,X^2}^{\lambda,\mu}$. It is a folklore that $\overline{\mathrm{Gr}}_{G,X^2}^{\lambda,\mu}$ gives a flat degeneration of $\overline{\mathrm{Gr}}_G^\lambda \times \overline{\mathrm{Gr}}_G^\mu$ to $\overline{\mathrm{Gr}}_G^{\lambda+\mu}$. As far as the author knows, no written proof is available. We will prove a slightly different version of this folklore in Proposition 1.2.4, using Theorem 1.2.2, which is originally due to [12, Theorem 1].

1.1.9. Assume that G is simple and that X is a complete curve in this subsection. Let $Bun_{G,X}$ be the moduli stack of principal G-bundles on X. Then there are natural morphisms $\pi : \mathrm{Gr}_G \to Bun_{G,X}$ and $\pi_n : \mathrm{Gr}_{G,X^n} \to Bun_{G,X}$ by simply forgetting the trivializations. First, assume that G is simply-connected. Then it is known that $Pic(Bun_{G,X}) \cong \mathbb{Z}$ (cf. [3]), and one of the generators $\mathcal{L}$ is ample. It is known that $\pi^*\mathcal{L}$ on Gr_G is just $\mathcal{L}_G$ defined in 1.1.2. Therefore, one also denotes the invertible sheaf $\pi_n^*\mathcal{L}$ on Gr_{G,X^n} by $\mathcal{L}_G$. It is clear from the definition that over Gr_{G,X^2}

$$\mathcal{L}_G \otimes \mathbb{C}_{(x,y)} \cong \begin{cases} \mathcal{L}_G \boxtimes \mathcal{L}_G \text{ on } \mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,y} & \text{if } x \neq y, \\ \mathcal{L}_G \text{ on } \mathrm{Gr}_{G,x} & \text{if } x = y \end{cases}$$

where $\mathbb{C}_{(x,y)}$ is the skyscraper sheaf at (x,y). Now if G is not simply-connected, we have: $\pi_0(\mathrm{Gr}_{G,X^2}) \cong \pi_1(G)$, and different connected components of Gr_{G,X^2} are isomorphic as ind-schemes. Denote $\mathcal{L}_G$ the invertible sheaf on Gr_{G,X^2}, the restrictions of which to each component are all isomorphic to the one on the neutral component, which is in turn isomorphic to $\mathcal{L}_{\tilde{G}}$, where $\tilde{G}$ is the simply-connected cover of G.

1.2. A flat degeneration of Schubert varieties

1.2.1. Let G be a simple group and $\tilde{G}$ be its simply-connected cover. Let k be a positive integer. Then $H^0(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G^{\otimes k})^*$ is a $\tilde{G}_{\mathcal{O}}$-module (and therefore a $\tilde{G}$-module), called the affine Demazure module (of level k). The first main theorem of [12] (Theorem 1) claims

1.2.2. Theorem. *$H^0(\overline{\mathrm{Gr}}_G^{\lambda+\mu}, \mathcal{L}_G^{\otimes k})^* \cong H^0(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G^{\otimes k})^* \otimes H^0(\overline{\mathrm{Gr}}_G^\mu, \mathcal{L}_G^{\otimes k})^*$ as $\tilde{G}$-modules.*

The proof of this theorem in [12] is of combinatoric flavor. Here we give a purely algebro-geometrical proof.

Proof. Fix a closed point $p\in X$. We consider the following ind-scheme over X.

$$\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}(R)=\left\{\begin{array}{l}x\in X(R),\ \mathcal{F}_1,\mathcal{F}_2\text{ two }G\text{-torsors on }X_R,\\ \beta_1\text{ a trivialization of }\mathcal{F}_1\text{ on }X_R-x,\beta_2\colon\ \mathcal{F}_2|_{(X-p)_R}\cong\mathcal{F}_1|_{(X-p)_R}\end{array}\right\}.$$

We have the projection

$$p:\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}\to\mathrm{Gr}_{G,X}$$

sending $(x,\mathcal{F}_1,\mathcal{F}_2,\beta_1,\beta_2)$ to $(x,\mathcal{F}_1,\beta_1)$. This map realizes $\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}$ as a fibration over Gr_X, with fibers isomorphic to $\mathrm{Gr}_{G,p}$.

Indeed, there is a $G_{\mathcal{O},p}$-torsor $\mathcal{P}$ over $\mathrm{Gr}_{G,X}$ given by

$$\mathcal{P}(R)=\left\{\begin{array}{l}x\in X(R),\ \mathcal{F}\text{ a }G\text{-torsora on }X_R,\\ \beta_1\text{ a trivialization of }\mathcal{F}\text{ on }X_R-x,\ \beta_2\text{ a trivialization of }\mathcal{F}\text{ on }\hat{p}\end{array}\right\}.$$

Then it is easy to see that $\mathcal{P}\times^{G_{\mathcal{O},p}}\mathrm{Gr}_{G,p}\cong\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}$. Observe that $\mathcal{P}|_{X-p}$ is a indeed a trivial $G_{\mathcal{O},p}$-torsor, with the section $\mathrm{Gr}_{G,X}|_{X-p}\to\mathcal{P}|_{X-p}$ given by $(x,\mathcal{F},\beta)\mapsto(x,\mathcal{F},\beta,\beta|_{\hat{p}})$. Therefore,

$$\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}|_{X-p}\cong\mathrm{Gr}_{G,X-p}\times\mathrm{Gr}_{G,p}.$$

On the other hand, it is clear

$$\mathrm{Gr}_{G,p}\tilde{\times}\mathrm{Gr}_{G,p}:=\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}|_p\cong G_{\mathcal{K}_p}\times^{G_{\mathcal{O}_p}}\mathrm{Gr}_{G,p}.$$

Now we denote

$$\overline{\mathrm{Gr}}^\lambda_{G,X}\tilde{\times}\overline{\mathrm{Gr}}^\mu_{G,p}=\mathcal{P}|_{\overline{\mathrm{Gr}}^\lambda_{G,X}}\times^{G_{\mathcal{O},p}}\overline{\mathrm{Gr}}^\mu_{G,p}.$$

This is a fibration over $\overline{\mathrm{Gr}}^\lambda_{G,X}$ with fibers isomorphic to $\overline{\mathrm{Gr}}^\mu_{G,p}$. Furthermore,

$$\overline{\mathrm{Gr}}^\lambda_{G,X}\tilde{\times}\overline{\mathrm{Gr}}^\mu_{G,p}|_{X-p}\cong\overline{\mathrm{Gr}}^\lambda_{G,X-p}\times\overline{\mathrm{Gr}}^\mu_{G,p}.$$

Since $\overline{\mathrm{Gr}}^\lambda_{G,X}$ is a fibration over X with fibers isomorphic to $\overline{\mathrm{Gr}}^\lambda_G$, we obtain that $\overline{\mathrm{Gr}}^\lambda_{G,X}\tilde{\times}\overline{\mathrm{Gr}}^\mu_{G,p}$ is flat over X, and that the special fiber

$$\overline{\mathrm{Gr}}^\lambda_{G,p}\tilde{\times}\overline{\mathrm{Gr}}^\mu_{G,p}:=\overline{\mathrm{Gr}}^\lambda_{G,X}\tilde{\times}\overline{\mathrm{Gr}}^\mu_{G,p}|_p$$

is a fibration over $\overline{\mathrm{Gr}}^\lambda_{G,p}$ with fibers isomorphic to $\overline{\mathrm{Gr}}^\mu_{G,p}$. In particular, the special fiber is reduced.

Observe that we have the natural map

$$m:\mathrm{Gr}_{G,X}\tilde{\times}\mathrm{Gr}_{G,p}\to\mathrm{Gr}_{G,X^2}|_{X\times p}$$

by sending $(x, \mathcal{F}_1, \mathcal{F}_2, \beta_1, \beta_2)$ to $(x, p, \mathcal{F}_2, \beta_1 \circ \beta_2)$. This is an isomorphism away from p and over p, it is just the usual convolution

$$m_p : G_{\mathcal{K}_p} \times^{G_{\mathcal{O}_p}} \mathrm{Gr}_{G,p} \to \mathrm{Gr}_{G,p}.$$

Observe that over the special fiber

$$m_p : \overline{\mathrm{Gr}}^{\lambda}_{G,p} \tilde{\times} \overline{\mathrm{Gr}}^{\mu}_{G,p} \to \overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}$$

is indeed a partial Bott–Samelson resolution.

Recall the invertible sheaf $\mathcal{L}_G^{\otimes k}$ on Gr_{G,X^2}. Then by 1.1.9, the restriction of $m^*\mathcal{L}_G^{\otimes k}$ to $\mathrm{Gr}_{G,X} \tilde{\times} \mathrm{Gr}_{G,p}|_{X-p} \cong \mathrm{Gr}_{G,X-p} \times \mathrm{Gr}_{G,p}$ is just $\mathcal{L}_G^{\otimes k} \boxtimes \mathcal{L}_G^{\otimes k}$. Since $\overline{\mathrm{Gr}}^{\lambda}_{G,X} \tilde{\times} \overline{\mathrm{Gr}}^{\mu}_{G,p}$ is a flat family over X, we obtains

$$H^0\big(\overline{\mathrm{Gr}}^{\lambda}_{G,x}, \mathcal{L}_G^{\otimes k}\big) \otimes H^0\big(\overline{\mathrm{Gr}}^{\mu}_{G,p}, \mathcal{L}_G^{\otimes k}\big) \cong H^0\big(\overline{\mathrm{Gr}}^{\lambda}_{G,p} \tilde{\times} \overline{\mathrm{Gr}}^{\mu}_{G,p}, m_p^*\mathcal{L}_G^{\otimes k}\big).$$

Finally, the theorem follows from the well-known fact that $H^0(\overline{\mathrm{Gr}}^{\lambda}_{G,p} \tilde{\times} \overline{\mathrm{Gr}}^{\mu}_{G,p}, m_p^*\mathcal{L}_G^{\otimes k}) \cong H^0(\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}, \mathcal{L}_G^{\otimes k})$ (e.g. [24, Theorem 2.16]). □

1.2.3. We still fix a point $p \in X$. Recall the variety $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X^2}$ from 1.1.8. Denote the reduced base change scheme by $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p} := (\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X^2}|_{X\times p})_{\mathrm{red}}$. We prove that

1.2.4. Proposition. *$\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is a scheme flat over $X \cong X \times p$, whose fiber over $x \neq p$ is $\overline{\mathrm{Gr}}^{\lambda}_{G,x} \times \overline{\mathrm{Gr}}^{\mu}_{G,p}$ and whose fiber over p is $\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}$.*

Proof. We give another interpretation of $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$. Recall the section $s^{\lambda,\mu}$ of Gr_{G,X^2} from 1.1.8. The restriction $s^{\lambda,\mu}|_{X\times p}$ gives a section of $\mathrm{Gr}_{G,X^2}|_{X\times p}$ over $X \times p$, also denoted by $s^{\lambda,\mu}$. The group scheme $\mathcal{G} := G_{\mathcal{O},X^2}|_{X\times p}$ acts on $\mathrm{Gr}_{G,X^2}|_{X\times p}$. Then $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is the closure of the orbit $\mathcal{G}s^{\lambda,\mu}$ in $\mathrm{Gr}_{G,X^2}|_{X\times p}$. In other words, $s^{\lambda,\mu}$ defines a morphism $\mathcal{G} \to \mathrm{Gr}_{G,X^2}|_{X\times p}$ by

$$\varphi : \mathcal{G} = \mathcal{G} \times_X X \xrightarrow{id\times s^{\lambda,\mu}} \mathcal{G} \times_X \mathrm{Gr}_{G,X^2}|_{X\times p} \to \mathrm{Gr}_{G,X^2}|_{X\times p}.$$

Then $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is the scheme-theoretic image of this morphism. That is, $\mathcal{O}_{\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}}$ is the image sheaf of the comorphism $\mathcal{O}_{\mathrm{Gr}_{G,X^2}|_{X\times p}} \to \varphi_*\mathcal{O}_{\mathcal{G}}$, and therefore, a subsheaf of $\varphi_*\mathcal{O}_{\mathcal{G}}$. Since $\mathcal{G}$ is formally smooth over X, the local parameter of X is not a zero divisor in $\varphi_*\mathcal{O}_{\mathcal{G}}$, nor in $\mathcal{O}_{\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}}$. This proves that $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is flat over X.

It is easy to see that

$$\big((\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_x\big)_{\mathrm{red}} = \begin{cases} \overline{\mathrm{Gr}}^{\lambda}_{G,x} \times \overline{\mathrm{Gr}}^{\mu}_{G,p} & x \neq p, \\ \overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p} & x = p. \end{cases}$$

Away from p, $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is étale locally trivial over $X-p$, and therefore, $(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_x = \overline{\mathrm{Gr}}^{\lambda}_{G,x} \times \overline{\mathrm{Gr}}^{\mu}_{G,p}$ for $x \neq p$. (Or if one assumes that $X = \mathbb{A}^1$, then the family is in fact trivial.) Now $\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}$ is the closed subscheme of $(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_p$ defined by the nilpotent radical. We have the exact sequence of sheaves

$$0 \to \mathcal{J} \to \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_p} \to \mathcal{O}_{\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}} \to 0.$$

Tensoring the ample invertible sheaf $\mathcal{L}_G^{\otimes k}$ constructed in 1.1.9, where k is sufficiently large, one obtains

$$0 \to H^0\big(\big(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}\big)_p, \mathcal{J} \otimes \mathcal{L}_G^{\otimes k}\big) \to H^0\big(\big(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}\big)_p, \mathcal{L}_G^{\otimes k}\big) \to H^0\big(\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}, \mathcal{L}_G^{\otimes k}\big) \to 0.$$

By the flatness of $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ and Theorem 1.2.2, $H^0((\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_p, \mathcal{L}_G^{\otimes k})$ and $H^0(\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p}, \mathcal{L}_G^{\otimes k})$ are both isomorphic to $H^0(\overline{\mathrm{Gr}}^{\lambda}_G, \mathcal{L}_G^{\otimes k}) \otimes H^0(\overline{\mathrm{Gr}}^{\mu}_G, \mathcal{L}_G^{\otimes k})$. Whence, $H^0((\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})_p, \mathcal{J} \otimes \mathcal{L}_G^{\otimes k}) = 0$. This implies $\mathcal{J} = 0$ since $\mathcal{J} \otimes \mathcal{L}_G^{\otimes k}$ are generated by global sections for k sufficiently large. □

1.3. T-fixed point subscheme of the affine Grassmannian

1.3.1. For any group scheme G over $\mathbb{C}$ operating on a presheaf Y on $\mathbf{Aff}_{\mathbb{C}}$, there is the notion of the fixed point subfunctor $Y^G \subset Y$ as defined in [8]

$$Y^G(R) := \big\{x \in Y(R) \,\big|\, \text{for all } R \to R',\ \text{and } g \in G(R'),\ gx_{R'} = x_{R'}\big\}$$

where $x_{R'}$ denote the image of x under the natural restriction $Y(R) \to Y(R')$. It is easy to check that if Y is a sheaf, then Y^G is a subsheaf. Something that is less trivial (cf. [8, II, Section 1, Theorem 3.6(d)]) is the following

1.3.2. Lemma. *If Y is a separated $\mathbb{C}$-scheme, then Y^G is a closed subscheme of Y.*

1.3.3. Now assume that G is a reductive algebraic group over $\mathbb{C}$. Let $\iota : L \to G$ be a Levi subgroup of some parabolic subgroup of G. Denote also by $\iota : \mathrm{Gr}_L \to \mathrm{Gr}_G$ the embedding induced from $\iota : L \to G$. Denote $H = Z(L)^0$ the neutral connected component of the center $Z(L)$ of L. Then H is a torus of G, and therefore acts on Gr_G. It is well known that the $H(\mathbb{C})$-fixed point set of $\mathrm{Gr}_G(\mathbb{C})$ is just $\mathrm{Gr}_L(\mathbb{C})$. The following theorem claims that this even holds scheme-theoretically.

1.3.4. Theorem. *The morphism $\iota : \mathrm{Gr}_L \to \mathrm{Gr}_G$ identifies Gr_L as the H-fixed point subsheaf of Gr_G.*

Proof. It is obvious that Gr_L is contained in $(\mathrm{Gr}_G)^H$. So we prove the converse. Observe that Gr_G is can be also obtained as the sheafification of the presheaf $R \mapsto G(R \hat{\otimes} \mathcal{K})/G(R \hat{\otimes} \mathcal{O})$ in the étale topology, it is enough to show for any R strictly Henselian local ring, $\mathrm{Gr}_L(R) \to (\mathrm{Gr}_G)^H(R)$ is surjective. We need the following lemma, whose proof is communicated to the author by Zhiwei Yun.

1.3.5. Lemma. *Let R be a $\mathbb{C}$-algebra. If Pic*(Spec R) $= 0$*, then*

$$G(R \hat{\otimes} \mathcal{K}) = B(R \hat{\otimes} \mathcal{K}) G(R \hat{\otimes} \mathcal{O}).$$

Proof. Let $X = G/B$ be the flag variety of G defined over $\mathbb{C}$. Then for any $\mathbb{C}$-algebra A, there is an injective map of sets $G(A)/B(A) \hookrightarrow X(A)$, since $G(A)/B(A)$ is the set of isomorphism classes of pairs $(\mathcal{F}_0, \beta)$, where $\mathcal{F}_0$ is a trivial B-torsor over Spec A and $\beta : \mathcal{F}_0 \to G$ is a B-equivariant morphism, whereas $X(A)$ is the set of isomorphism classes of pairs $(\mathcal{F}, \beta)$, where $\mathcal{F}$ a B-torsor over Spec A and $\beta : \mathcal{F} \to G$ is a B-equivariant morphism.

Now let $F = R \hat{\otimes} \mathcal{K}$, $S = R \hat{\otimes} \mathcal{O}$. Then one has $X(F) = X(S)$ because X is projective and one can always make an S-point out of an F-point by multiplying a common denominator of coordinates. Therefore, the lemma holds if $G(S)/B(S) = X(S)$. Clearly from previous description, it is enough to show that $H^1(\operatorname{Spec} S, B) = 0$. Observe that B can be filtered by normal subgroups such that the associated quotients are $\mathbb{G}_m$s and $\mathbb{G}_a$s. Since $\mathbb{G}_a$ is a coherent sheaf, $H^1(\operatorname{Spec} S, \mathbb{G}_a)$ is always trivial. To show $H^1(\operatorname{Spec} S, \mathbb{G}_m) = 0$, observe that any $\mathbb{G}_m$-torsor $\mathcal{F}$ over Spec S has a section when restricted to Spec $R \hookrightarrow$ Spec S since *Pic*(Spec R) $= 0$. Now $\mathcal{F}$ is smooth over Spec S, so the section of $\mathcal{F}|_{\operatorname{Spec} R}$ extends to a section of $\mathcal{F}$ over Spec S. Therefore, the lemma follows. $\square$

Conclusion of the proof of the theorem. Fix a uniformizer $t \in \mathcal{O}$ for convenience. Let P be a parabolic subgroup of G containing L. By above lemma $G(R((t))) = P(R((t)))G(R[\![t]\!])$ since R is strictly Henselian. Now let $x \in (\mathrm{Gr}_G)^H(R)$, one could assume that x is represented by an element $g \in P(R((t)))$. Then x being a H-fixed point is equivalent to $g^{-1}tg \in P(R[\![t]\!])$ for any $t \in H(R)$. Let U_P be unipotent radical of P. Since $L \times U_P \to P$ is an isomorphism (as $\mathbb{C}$-varieties), one could write $g = g'g''$ with $g' \in L(R((t)))$, $g'' \in U_P(R((t)))$. Furthermore, since U_P is unipotent, g'' could be uniquely written as $g'' = g_1g_2$ with $g_1 \in U_P(t^{-1}R[t^{-1}])$ and $g_2 \in U_P(R[\![t]\!])$. Then $g^{-1}tg = g_2^{-1}g_1^{-1}tg_1g_2 \in P(R[\![t]\!])$ is equivalent $g_1 = tg_1t^{-1}$ for any $t \in H(R)$. Therefore $g_1 = 1$, and x is represented by an element $g \in L(R((t)))G(R[\![t]\!])$, that is $x \in \mathrm{Gr}_L(R)$. $\square$

1.3.6. Corollary. *Let $\iota : T \to G$ be a maximal torus of G. Then $\iota : \mathrm{Gr}_T \to \mathrm{Gr}_G$ identifies Gr_T as the T-fixed point subsheaf of Gr_G.*

1.3.7. Let $M = [L, L]$ be the derived group of a Levi subgroup L of G. Then M is a semisimple subgroup of G. Let $T_M \subset T_G$ be the maximal torus of M and G. For any $s^\lambda \in \mathrm{Gr}_G$, the morphism $M_{\mathcal{K}} \to \mathrm{Gr}_G$ given by $m \to m \cdot s^\lambda$ induces a closed embedding of $\mathrm{Gr}_M \to \mathrm{Gr}_G$ by [26, Lemma 3.2]. It is easy to see that Gr_M is T_G-invariant. We have another corollary of Theorem 1.3.4.

1.3.8. Corollary. *Under the above assumption, $(\mathrm{Gr}_M)^{T_G} \cong (\mathrm{Gr}_M)^{T_M}$.*

Proof. Recall the natural embedding $\mathrm{Gr}_L \subset \mathrm{Gr}_G$. For simplicity, we assume that G is simply-connected. (The general case is very similar.) Then M is simply-connected and therefore, $\Lambda_M = R_M$. The connected components of Gr_L are labelled by elements in Λ_G/Λ_M. Then the way of embedding of Gr_M into Gr_G realized Gr_M as the reduced subscheme of the component of Gr_L corresponding to λ mod Λ_M. Observe that $T_G = T_M \cdot Z(L)^0$, where $Z(L)^0$ is the neutral component of the center of L. Therefore, $(\mathrm{Gr}_M)^{T_G} \cong ((\mathrm{Gr}_M)^{Z(L)^0})^{T_M} = (\mathrm{Gr}_M)^{T_M}$. $\square$

1.4. The Borel–Weil theorem

Assume that G is a simple algebraic group in this section. Let $\mathfrak{g}$ be its Lie algebra, and $\hat{\mathfrak{g}}$ be the untwisted affine algebra associated to $\mathfrak{g}$.

1.4.1. Let Gr_G be the affine Grassmannian of G, and $\mathcal{L}_G$ an invertible sheaf on Gr_G, which is the positive generator of the Picard group of each connected component. (See Theorem 1.1.2(3) and (4).) It is well known that $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)$ has a natural $\hat{\mathfrak{g}}$-module structure.

1.4.2. Recall that the irreducible integrable representations of $\hat{\mathfrak{g}}$ are parameterized by $(k, \check{\nu})$, where k is a positive integer, called the level, and $\check{\nu}$ is a dominant integral weight of $\mathfrak{g}$ such that $(\check{\theta}, \check{\nu})^* \leqslant k$. For such $(k, \check{\nu})$, the irreducible integrable representation of highest weight $(k\Lambda + \check{\nu})$ is denoted by $L(k\Lambda + \check{\nu})$.

1.4.3. Recall that a fundamental coweight ω_i of $\mathfrak{g}$ is called minuscule if $\langle \omega_i, \check{\alpha} \rangle \leqslant 1$ for any $\check{\alpha}$ positive root of $\mathfrak{g}$. We will also include the zero coweight as a minuscule coweight. It can be shown (for example, see [7, Chapter VI, exercises]) that for any $\gamma \in \pi_1(G) \cong \Lambda_G/R_G, \gamma \neq 0$, there is a unique $i_\gamma \in I$, such that: (i) $\omega_{i_\gamma} \in \Lambda_G$; (ii) ω_{i_γ} mod $R_G = \gamma$; (iii) ω_{i_γ} is a minuscule coweight. If $\gamma = 0$, we set the corresponding minuscule coweight $\omega_{i_0} = 0$.

We have the following Borel–Weil theorem.

1.4.4. Proposition. *Assumptions are as above. One has*

$$\Gamma\big(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k}\big)^* \cong \bigoplus_{\gamma \in \pi_1(G)} L(k\Lambda + k\iota\omega_{i_\gamma})$$

as $\hat{\mathfrak{g}}$*-modules.*

Proof. Choose $T \subset G$ a maximal torus. Let $\tilde{G} \to G$ be the simply-connected cover of G. Then $\pi_1(G) \cong \Lambda_G/R_G$ is also regarded as a subgroup of $\tilde{G}$. For any $\gamma \in \pi_1(G)$, choose $\lambda_\gamma \in \Lambda_G$ a lifting of γ, e.g. $\lambda_\gamma = \omega_{i_\gamma}$. One also fixes a uniformizer $t \in \mathcal{O}$. From the exact sequence of étale sheaves on $\operatorname{Spec} \mathcal{K}$

$$1 \to \pi_1(G) \to \tilde{G} \to G \to 1$$

one obtains that

$$0 \to \pi_1(G) \to \tilde{G}_{\mathcal{K}} \to G_{\mathcal{K}} \to H^1\big(\operatorname{Spec} \mathcal{K}, \pi_1(G)\big) \cong \pi_1(G) \to 0$$

and therefore

$$G_{\mathcal{K}} = \bigsqcup_{\gamma \in \pi_1(G)} t^{\lambda_\gamma}\big(\tilde{G}_{\mathcal{K}}/\pi_1(G)\big).$$

Since $t^\lambda \in G(\mathcal{K})$, Ad_{t^λ} acts on $\tilde{G}_{\mathcal{K}}/\pi_1(G) = G_{\mathcal{K}}^0$ by adjoint action. It is liftable to a unique automorphism of $\tilde{G}_{\mathcal{K}}$, also denoted by Ad_{t^λ}. Now

$$\mathrm{Gr}_G = \bigsqcup_{\gamma\in\pi_1(G)} (\mathrm{Gr}_G)^\gamma = \bigsqcup_{\gamma\in\pi_1(G)} \tilde{G}_{\mathcal{K}}/Ad_{t^{\lambda_\gamma}}\tilde{G}_{\mathcal{O}}.$$

Observe $(\mathrm{Gr}_G)^\gamma = \tilde{G}_{\mathcal{K}}/Ad_{t^{\lambda_\gamma}}\tilde{G}_{\mathcal{O}}$ for different γ's are isomorphic as ind-varieties, but not as $\tilde{G}_{\mathcal{K}}$-homogeneous spaces. Indeed, $Ad_{t^{\lambda_\gamma}}\tilde{G}_{\mathcal{O}}$ is the hyperspecial parahoric subgroup in $\tilde{G}_{\mathcal{K}}$ obtained by deleting the vertex i_γ of the affine Dynkin diagram. Therefore, $\Gamma((\mathrm{Gr}_G)^\gamma, \mathcal{L}_G^{\otimes k})^* \cong L(k\Lambda + k\check{\omega}_{i_\gamma})$ as $\hat{\mathfrak{g}}$-modules by [24] and [27]. Remark that in [24] and [27], the definition of the (partial) flag variety is different from the definition here. However, the two definitions are the same due to [1, Theorem 7.7]. Since $\check{\alpha}_{i_\gamma}$ is always a long root for $\gamma \neq 0$, $\iota\alpha_{i_\gamma} = 2\check{\alpha}_{i_\gamma}/(\check{\alpha}_{i_\gamma}, \check{\alpha}_{i_\gamma})^* = \check{\alpha}_{i_\gamma}$. And therefore, $\check{\omega}_{i_\gamma} = \iota\omega_{i_\gamma}$. $\square$

2. Proof of Theorem 0.2.2

In this section, we will prove the main theorem. Our strategy is first to reduce the full theorem to some special cases, where the geometry of Schubert varieties is simple, and then to prove these special cases. We will also discuss the smooth locus of Schubert varieties in this section.

2.1. First reductions

Let G be a simple algebraic group, which is not assumed to be simply-laced in this section. We will continue using the notations introduced in Section 1. Recall that $(\overline{\mathrm{Gr}}_G^\lambda)^T$ is a finite scheme supported at $\{s^{w\mu} \mid \mu \in \Lambda_G^+, \mu \leqslant \lambda, w \in W\}$. Let $\mathcal{I}^\lambda \subset \mathcal{O}_{\overline{\mathrm{Gr}}_G^\lambda}$ be the ideal sheaf defining $(\overline{\mathrm{Gr}}_G^\lambda)^T$. For any quasi-coherent sheaf $\mathcal{F}$ on Gr_G, we will denote $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}_G^{\otimes n}$. We prove that

2.1.1. Proposition. *For k any positive integer, the natural morphism $\mathcal{L}_G^{\otimes k} \to \mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T} \otimes \mathcal{L}_G^{\otimes k}$ induces a surjective map*

$$\Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G^{\otimes k}\big) \twoheadrightarrow \Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T} \otimes \mathcal{L}_G^{\otimes k}\big).$$

Proof. By Corollary 1.3.6, the natural embedding $\mathrm{Gr}_T \to \mathrm{Gr}_G$ identifies Gr_T as the T-fixed subscheme of Gr_G. Therefore,

$$\Gamma\big(\mathrm{Gr}_T, \mathcal{L}_G^{\otimes k}\big) = \varprojlim \Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T} \otimes \mathcal{L}_G^{\otimes k}\big)$$

where the limit is taken over $\lambda \in \Lambda_G^+$. On the other hand, we have

$$\Gamma\big(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k}\big) = \varprojlim \Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G^{\otimes k}\big).$$

Since for $\lambda \geqslant \mu$, $(\overline{\mathrm{Gr}}_G^\mu)^T$ is a closed subscheme of $(\overline{\mathrm{Gr}}_G^\lambda)^T$ and both are finite schemes, $\Gamma((\overline{\mathrm{Gr}}_G^\lambda)^T, \mathcal{L}_G^{\otimes k}) \to \Gamma((\overline{\mathrm{Gr}}_G^\mu)^T, \mathcal{L}_G^{\otimes k})$ is surjective. Therefore, to prove the proposition, it is enough to show that the natural map

$$\Gamma\big(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k}\big) \to \Gamma\big(\mathrm{Gr}_T, \mathcal{L}_G^{\otimes k}\big)$$

is surjective, or equivalently,

$$\Gamma\big(\mathrm{Gr}_T,\mathcal{L}_G^{\otimes k}\big)^* \to \Gamma\big(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k}\big)^*$$

is injective.

Observe that both $\Gamma(\mathrm{Gr}_T,\mathcal{L}_G^{\otimes k})^*$ and $\Gamma(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k})^*$ have module structures over the Heisenberg Lie algebra $\hat{\mathfrak{t}}\subset\hat{\mathfrak{g}}$, which is the restriction of the central extension of $\mathfrak{g}\hat{\otimes}\mathcal{K}$ to $\mathfrak{t}\hat{\otimes}\mathcal{K}$ (see 3.1.1). Furthermore, the map $\Gamma(\mathrm{Gr}_T,\mathcal{L}_G^{\otimes k})^*\to\Gamma(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k})^*$ is a $\hat{\mathfrak{t}}$-mod morphism. Since Gr_T is discrete,

$$\Gamma\big(\mathrm{Gr}_T,\mathcal{L}_G^{\otimes k}\big)=\bigoplus_{\lambda\in\Lambda_G}\mathcal{O}_{\mathrm{Gr}_T,s^\lambda}\otimes\mathcal{L}_G^{\otimes k}\big|_{s^\lambda}$$

and 1.1.3 implies that as $\hat{\mathfrak{t}}$-modules, $(\mathcal{O}_{\mathrm{Gr}_T,s^\lambda}\otimes\mathcal{L}_G^{\otimes k}|_{s^\lambda})^*\cong\pi^k_{-k\lambda}$ is the Fock $\hat{\mathfrak{t}}$-module as defined in 3.1.1.

To prove the proposition, it is enough to show that the map $\pi^k_{-k\lambda}\to\Gamma(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k})^*$ is not zero. Then it must be injective since $\pi^k_{-k\lambda}$ is simple. Dually, it is enough to show

$$\Gamma\big(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k}\big)\to\mathcal{O}_{\mathrm{Gr}_T,s^\lambda}\otimes\mathcal{L}_G^{\otimes k}\big|_{s^\lambda}$$

is not zero. However, this is clear. For any λ, let σ_λ be the linear form on $\bigoplus_{\gamma\in\pi_1(G)}L(\Lambda+\iota\omega_{i_\gamma})$, which is not zero along $t^\lambda\cdot v_\Lambda$, and zero on any other weight vectors, where t is some chosen uniformizer, and v_Λ is the line of highest weight vectors in $L(\Lambda)$. (See the proof of Lemma 2.2.2 for the meaning of $t^\lambda\cdot v_\Lambda$.) Then $\sigma_\lambda^k\in\Gamma(\mathrm{Gr}_G,\mathcal{L}_G^{\otimes k})$ maps nonzero to $\mathcal{O}_{\mathrm{Gr}_T,s^\lambda}\otimes\mathcal{L}_G^{\otimes k}|_{s^\lambda}$. $\square$

Therefore, Theorem 0.2.2 follows from the following

2.1.2. Theorem. *Assume that G is a simple algebraic group of type A or D. Then for any $\lambda\in\Lambda_G$, $\Gamma(\overline{\mathrm{Gr}}_G^\lambda,\mathcal{I}^\lambda(1))=0$.*

This theorem is also proved for certain λ if G is of type E (see Sections 2.2.16, 2.2.17). We expect it holds for all λ in this case.

We next show that, to prove Theorem 2.1.2, it is enough to prove it for fundamental coweights.

2.1.3. Proposition. *Let G be a simple, but not necessarily simply-laced algebraic group. For $\lambda,\mu\in\Lambda_G^+$, if $\Gamma(\overline{\mathrm{Gr}}_G^\lambda,\mathcal{I}^\lambda(1))=\Gamma(\overline{\mathrm{Gr}}_G^\mu,\mathcal{I}^\mu(1))=0$, then $\Gamma(\overline{\mathrm{Gr}}_G^{\lambda+\mu},\mathcal{I}^{\lambda+\mu}(1))=0$.*

Proof. Recall the variety $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ from 1.2.3. T acts on $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ by embedding $T_{X\times p}\hookrightarrow\mathcal{G}=G_{\mathcal{O},X^2}|_{X\times p}$. Let $(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})^T$ be the T-fixed subscheme of $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$. Then its fiber over $x\neq p$ is $(\overline{\mathrm{Gr}}^\lambda_{G,x})^T\times(\overline{\mathrm{Gr}}^\mu_{G,p})^T$ and over p is $(\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p})^T$.

For simplicity, we will assume that $X=\mathbb{A}^1$ now. Then we claim that $(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})^T$ is flat over $X-p$. To see this, one regards $X-p\cong\mathbb{G}_m$ so that $\mathbb{G}_m$ acts on $X-p$. It is easy to see there is a $\mathbb{G}_m$-action on $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}|_{X-p}$, commuting with the T-action, such that the natural projection $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}|_{X-p}\to X-p$ is an equivariant map. Therefore, $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}|_{X-p}\cong(X-p)\times(\overline{\mathrm{Gr}}^\lambda_G\times\overline{\mathrm{Gr}}^\mu_G)$ and $(\overline{\mathrm{Gr}}^{\lambda,\mu}_G)^T|_{X-p}\cong(X-p)\times((\overline{\mathrm{Gr}}^\lambda_G)^T\times(\overline{\mathrm{Gr}}^\mu_G)^T)$. Our claim follows. Now let Z be the *scheme-theoretical closure* of $(\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})^T|_{X-p}\hookrightarrow\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ (cf. [18, II Ex. 3.11(d), p. 92]). Z is a closed subscheme of $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$. Since $\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p}$ is flat over X (cf. Proposition 1.2.4), so is Z (cf. [18, III, Proposition 9.8, p. 258]). Certainly, $Z \subset (\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})^T$. Therefore, $Z_p \subset (\overline{\mathrm{Gr}}^{\lambda,\mu}_{G,X\times p})^T|_p \cong (\overline{\mathrm{Gr}}^{\lambda+\mu}_{G,p})^T$. Since both of them are finite schemes, we have

$$\begin{aligned}
&\dim \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda+\mu}_G)^T}\\
&\quad \geqslant \dim \mathcal{O}_{Z_p}\\
&\quad = \dim \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda}_G)^T} \dim \mathcal{O}_{(\overline{\mathrm{Gr}}^{\mu}_G)^T} && \text{(by the flatness of } Z\text{)}\\
&\quad = \dim H^0\big(\overline{\mathrm{Gr}}^{\lambda}_G, \mathcal{L}_G\big) \dim H^0\big(\overline{\mathrm{Gr}}^{\mu}_G, \mathcal{L}_G\big) && \text{(by the assumption of the lemma)}\\
&\quad = \dim H^0\big(\overline{\mathrm{Gr}}^{\lambda+\mu}_G, \mathcal{L}_G\big) && \text{(by Theorem 1.2.2)}\\
&\quad \geqslant \dim \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda+\mu}_G)^T} && \text{(by Proposition 2.1.1).}
\end{aligned}$$

Therefore, $\dim H^0(\overline{\mathrm{Gr}}^{\lambda+\mu}_G, \mathcal{L}_G) = \dim \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda+\mu}_G)^T}$. Since

$$H^0\big(\overline{\mathrm{Gr}}^{\lambda+\mu}_G, \mathcal{L}_G\big) \twoheadrightarrow H^0\big(\overline{\mathrm{Gr}}^{\lambda+\mu}_G, \mathcal{O}_{(\overline{\mathrm{Gr}}^{\lambda+\mu}_G)^T} \otimes \mathcal{L}_G\big)$$

is a surjective by Proposition 2.1.1, and they have the same dimension, the map must be an isomorphism. □

2.2. Proof of Theorem 2.1.2

By Proposition 2.1.3, it is enough to prove the Theorem 2.1.2 for λ being a fundamental coweight. Therefore, we could assume from now on that G is of adjoint type. However, G is not assumed to be simply-laced through 2.2.1–2.2.7.

2.2.1. Assume that G is a simple algebraic group. For $\check{\psi} = n\check{\delta} + \check{\alpha}$ a real root of $\hat{\mathfrak{g}}$, let $U_{\check{\psi}}$ be the corresponding unipotent subgroup in $G_{\mathcal{K}}$, i.e., $\operatorname{Lie} U_{\check{\psi}} = \mathbb{C}e_{n\check{\delta}+\check{\alpha}}$, where $e_{n\check{\delta}+\check{\alpha}}$ is a root vector of $n\check{\delta} + \check{\alpha}$. Let $S_{\check{\psi}} \subset G_{\mathcal{K}}$ be the subgroup generated by $U_{\check{\psi}}, U_{-\check{\psi}}$. For $\lambda \in \Lambda_G$, let s^λ be the corresponding point in Gr_G. For any choice of the uniformizer t, recall $t^\lambda \in G_{\mathcal{K}}$. Since

$$t^{-\lambda} U_{n\check{\delta}+\check{\alpha}} t^{\lambda} = U_{(n-\langle\check{\alpha},\lambda\rangle)\check{\delta}+\check{\alpha}}.$$

One obtains that $U_{n\check{\delta}+\check{\alpha}} s^\lambda \neq s^\lambda$ if and only if $n < \langle\check{\alpha}, \lambda\rangle$.

2.2.2. Lemma. *Under above assumptions, $S_{\check{\psi}} \cdot s^\lambda \subset \mathrm{Gr}_G$ is a T-invariant rational curve on Gr_G. There are two T-fixed points in $S_{\check{\psi}} \cdot s^\lambda$; namely, s^λ and $s^{\lambda-(\langle\lambda,\check{\alpha}\rangle-n)\alpha}$. Furthermore, the restriction of $\mathcal{L}_G$ to this rational curve is $\mathcal{O}(\frac{2(\langle\lambda,\check{\alpha}\rangle-n)}{(\check{\alpha},\check{\alpha})^*})$.*

In particular, if $\check{\alpha}$ is a long root, and $\langle\lambda, \check{\alpha}\rangle = 1$, then the restriction of $\mathcal{L}_G$ on $S_{\check{\alpha}} s^\lambda$ is of degree one.

2.2.3. Remark. In the proof of Theorem 2.1.2, we need to use a lot of rational curves of degree one (or equivalently, projective lines) in $\overline{\mathrm{Gr}}_G^{\omega_i}$. If G is not simply-laced, then for short root $\check{\alpha}$, $S_{\check{\alpha}}s^\lambda$ will not be of degree one, and there will not be enough projective lines for our purpose.

Proof. Since $\mathcal{L}_G$ is very ample, we use it to embed Gr_G into projective spaces. Recall from Proposition 1.4.4,

$$H^0(\mathrm{Gr}_G, \mathcal{L}_G)^* = \bigoplus_{\gamma\in\pi_1(G)} L(\Lambda + \iota\omega_{i_\gamma}).$$

This representation of $\hat{\mathfrak{g}}$ can be integrated to a projective action of $G_{\mathcal{K}}$ on it. More precisely, let $\tilde{G}$ be the simply-connected cover of G. Since for any $\lambda \in \Lambda_G$, Ad_{t^λ} acts on $\tilde{G}_{\mathcal{K}}$, one can define

$$\widetilde{G_{\mathcal{K}}} = \tilde{G}_{\mathcal{K}} \rtimes \Lambda_G / \tilde{G}_{\mathcal{K}} \rtimes R_G.$$

It follows from the construction that the neutral component $\widetilde{G_{\mathcal{K}}}^0$ of $\widetilde{G_{\mathcal{K}}}$ is isomorphic to $\tilde{G}_{\mathcal{K}}$ and $\widetilde{G_{\mathcal{K}}}$ is a $\pi_1(G)$-covering of $G_{\mathcal{K}}$. Then similarly to the simply-connected case as observed by Faltings, and explained in [25, Proposition 4.3], the $\hat{\mathfrak{g}}$-module $\bigoplus_{\gamma\in\pi_1(G)} L(\Lambda + \iota\omega_{i_\gamma})$ is integrated to a module over the $\mathbb{G}_m$-central extension of $\widetilde{G_{\mathcal{K}}}$. Let v_Λ be the line of highest weight vectors in $L(\Lambda)$. Then $\mathrm{Gr}_G = \widetilde{G_{\mathcal{K}}}/\tilde{G}_{\mathcal{O}}$ is embedded in $\mathbb{P}(\bigoplus_{\gamma\in\pi_1(G)} L(\Lambda + \iota\omega_{i_\gamma}))$ by $g \mapsto gv_\Lambda$ for $g \in \widetilde{G_{\mathcal{K}}}$. We recall the weight for the line $s^\mu = t^\mu \cdot v_\Lambda$ is $\Lambda - \iota\mu - \frac{(\mu,\mu)}{2}\check{\delta}$ (cf. [29, Proposition 3.1]).

Now observe that $S_{\check{\psi}}$ is isomorphic to SL_2 or PSL_2. We claim

Claim. *The Lie algebra of $S_{\check{\psi}}$ is the $\mathfrak{sl}_2$-triple spanned by*

$$\left\{e_{n\check{\delta}+\check{\alpha}},\ 2nK/(\check{\alpha},\check{\alpha})^* + \alpha, e_{-n\check{\delta}-\check{\alpha}}\right\}.$$

Assume this claim for the moment. Since $n < \langle\lambda, \check{\alpha}\rangle$, $U_{-\check{\psi}}s^\lambda = s^\lambda$. Therefore, $S_{\check{\psi}}s^\lambda \cong S_{\check{\psi}}/B_- \cong \mathbb{P}^1$, where B_- is the Borel subgroup of $S_{\check{\psi}}$ containing $U_{-\check{\psi}}$.

Let $V \subset L(\Lambda)$ be the minimal $\mathfrak{sl}_2$-stable subspace generated by $t^\lambda v_\Lambda$. Then V is an irreducible representation of this $\mathfrak{sl}_2$ of lowest weight

$$\left\langle \frac{2nK}{(\check{\alpha},\check{\alpha})^*} + \alpha, \Lambda - \iota\lambda - \frac{(\lambda,\lambda)}{2}\check{\delta} \right\rangle = \frac{2(n - \langle\lambda,\check{\alpha}\rangle)}{(\check{\alpha},\check{\alpha})^*}.$$

Now SL_2-theory shows that $S_{\check{\psi}}s^\lambda$ is contained in $\mathbb{P}(V)$, with two $(2nK/(\check{\alpha},\check{\alpha})^* + \alpha)$-fixed points s^λ and $s^{\lambda-(\langle\lambda,\check{\alpha}\rangle-n)\alpha}$. Furthermore, $S_{\check{\psi}}s^\lambda$ is rational in $\mathbb{P}(V)$ of degree $\frac{2(\langle\lambda,\check{\alpha}\rangle-n)}{(\check{\alpha},\check{\alpha})^*}$. □

It remains to prove the claim above. It is a direct consequence of the following lemma, which is well known. We include a proof for completeness.

2.2.4. Lemma. *Let $\check{\psi} = n\check{\delta} + \check{\alpha}$ be a real root, then the corresponding coroot is $\psi = \frac{2n}{(\check{\alpha},\check{\alpha})^*}K + \alpha$.*

Proof. Observe we could assume $n \geqslant 0$. The coroot corresponding to the simple root $\check{\alpha}_0 = \check{\delta} - \check{\theta}$ is $\alpha_0 = K - \theta$. We prove the lemma by induction on n.

For $n = 0$, it is clear. Assuming for $n - 1$, the lemma holds. Observe that since $\check{\theta}$ is a long root, then for any root $\check{\alpha} \neq \check{\theta}$, $|\langle\theta, \check{\alpha}\rangle| \leqslant 1$. If $\langle\theta, \check{\alpha}\rangle$=1, then $r_0((n-1)\check{\delta} + \check{\alpha}) = n\check{\delta} + \check{\alpha} - \check{\theta}$. where r_0 is the simple reflection in the affine Weyl group corresponding to the vertex i_0 in the affine Dynkin diagram. So the coroot corresponding to $n\check{\delta} + \check{\alpha} - \check{\theta}$ is

$$r_0\left(\frac{2(n-1)K}{(\check{\alpha}, \check{\alpha})^*} + \alpha\right) = \frac{2nK}{(\check{\alpha}, \check{\alpha})^*} + \alpha - \frac{2\theta}{(\check{\alpha}, \check{\alpha})^*}.$$

Observe that the coroot corresponding to $\check{\alpha} - \check{\theta}$ is exactly $\alpha - \frac{2\theta}{(\check{\alpha}, \check{\alpha})^*}$. Therefore, for $|\langle\theta, \check{\alpha}\rangle| = 1$, the coroot corresponding to $n\check{\delta} + \check{\alpha}$ is $2nK/(\check{\alpha}, \check{\alpha})^* + \alpha$. Now the lemma follows from the fact that for any $\check{\alpha} \in \Delta$, there exist $w \in W$ and $\check{\beta} \in \Delta$, $|\langle\theta, \check{\beta}\rangle| = 1$, such that $\check{\alpha} = w(\check{\beta})$. □

2.2.5. Let L be a Levi subgroup of some parabolic subgroup of G, and $M = [L, L]$ be the derived group of L. Let $T_M \to T_G = T_L$ be the maximal tori of M, G and L. Then there is a natural embedding of lattices $\Lambda_M \subset \Lambda_G$. In addition, we have a projection $p : \Lambda_G \to \Lambda_M$ defined as follows. Let $Z(L)^0$ be the neutral connected component of the center of L. Since G is assumed to be of adjoint-type, M is also of adjoint type. Therefore $T_M \times Z(L)^0 \to T_G$ is an isomorphism. Then the projection p is induced from $T_G \cong T_M \times Z(L)^0 \to T_M$.

For $\lambda \in \Lambda_G$, let $\mathrm{Gr}_M \to \mathrm{Gr}_G$ be the embedding, induced from $M_{\mathcal{K}} \to \mathrm{Gr}_G$ by sending $m \mapsto ms^\lambda$. We have the following

2.2.6. Lemma. *Under the above embedding, $\overline{\mathrm{Gr}}_M^{p(\lambda)} \subset \mathrm{Gr}_M \cap \overline{\mathrm{Gr}}_G^\lambda$. If the Dynkin subdiagram for M contains some vertex i of the Dynkin diagram of G corresponding to a long root $\check{\alpha}_i$ of G, then the restriction of $\mathcal{L}_G$ to Gr_M is isomorphic to $\mathcal{L}_M$.*

Proof. Let us prove the second statement. Without loss of generality, we could assume M is the derived group of a standard Levi factor in G. Let $\mu = \lambda - p(\lambda) + \omega_i$. Then $s^\mu \in \mathrm{Gr}_M$, and $S_{\check{\alpha}_i} s^\mu \subset \mathrm{Gr}_M$. Furthermore, the restriction of $\mathcal{L}_G$ to the rational curve $S_{\check{\alpha}_i} s^\mu$ is $\mathcal{O}(1)$ by Lemma 2.2.2, since $\check{\alpha}_i$ is a long root. Therefore, the restriction of $\mathcal{L}_G$ to Gr_M must be $\mathcal{L}_M$. □

2.2.7. Recall the definition of σ_λ from the proof of Proposition 2.1.1. Define $U_\lambda^\mu = \overline{\mathrm{Gr}}_G^\lambda - \mathrm{div}(\sigma_\mu)$. It is clear that U_λ^μ is an affine open subscheme of $\overline{\mathrm{Gr}}_G^\lambda$ containing s^μ as the unique T-fixed point. Therefore, $\{U_\lambda^\mu\}$ for all μ such that $s^\mu \in \overline{\mathrm{Gr}}_G^\lambda$ form an open cover of $\overline{\mathrm{Gr}}_G^\lambda$, for the complement of the union of these open subsets is closed without any T-fixed point and thus must be an empty set. We will denote U_λ^λ simply by U^λ. Let N^λ be the unipotent subgroup of $G_{\mathcal{O}}$ generated by $U_{n\check{\delta}+\check{\alpha}}$ with $\check{\alpha}$ positive root and $0 \leqslant n < \langle\lambda, \check{\alpha}\rangle$. Then $N^\lambda \cdot s^\lambda = U^\lambda$.

We will call an m-dimensional T-invariant subvariety Z in $\overline{\mathrm{Gr}}_G^\lambda$ admissible, if there exists some affine open subset of Z, which contains some open subset of $U_{n_1\check{\delta}+\check{\alpha}_{i_1}} \cdots U_{n_m\check{\delta}+\check{\alpha}_{i_m}} s^\mu$, where $s^\mu \in \overline{\mathrm{Gr}}_G^\lambda$, $n_1\check{\delta} + \check{\alpha}_{i_1}, \ldots, n_m\check{\delta} + \check{\alpha}_{i_m}$ are distinct, and $n_j \geqslant 0$. Observe that any T-invariant curve in Gr_G^λ is admissible. Furthermore, U^λ is admissible.

From now on, we assume that G is simple simply-laced algebraic group. We will prove Theorem 2.1.2 by case by case considerations. We embed Gr_G into a projective space by $\mathcal{L}_G$. The way we label the Dynkin diagram follows Bourbaki's notation (cf. [7, pp. 250–275]).

2.2.8. Proposition. *Theorem* 2.1.2 *holds if* λ *is a minuscule coweight.*

Proof. We shall point out that the proof presented here is not the simplest one. The reason, however, that we adapt it here is two fold. First, the proof itself is self-contained, and requires no knowledge about minuscule representations. Secondly and more importantly, it serves as a prototype for the discussions for non-minuscule fundamental coweights.

Recall that if λ is minuscule, then $\langle\lambda,\check{\alpha}\rangle \leqslant 1$ for any $\check{\alpha}$ positive root. Denote $\Delta_\lambda = \{\check{\alpha}\in\Delta_+, \langle\lambda,\check{\alpha}\rangle = 1\}$. In this case $\overline{\mathrm{Gr}}_G^\lambda = \mathrm{Gr}_G^\lambda = G/P_\lambda$, where P_λ is the parabolic subgroup corresponding to λ, i.e. P_λ is generated by B^- and $U_{\check{\alpha}}$ for $\check{\alpha}$ positive root not contained in Δ_λ. Let $\sigma\in H^0(\mathrm{Gr}_G^\lambda, \mathcal{I}^\lambda(1))$, then σ vanishes on all T-fixed points s^μ, with $\mu = w\lambda$ for some $w\in W$. For those $\check{\alpha}$ such that $\langle\lambda,\check{\alpha}\rangle = 1$, by Lemma 2.2.2, $S_{\check{\alpha}}\cdot s^\lambda$ is a rational curve, with two T-fixed points s^λ and $s^{\lambda-\alpha}$. Since the restriction of $\mathcal{L}_G$ on $S_{\check{\alpha}}\cdot s^\lambda$ is isomorphic to $\mathcal{O}(1)$ and σ vanishes at s^λ and $s^{\lambda-\alpha}$, $\sigma|_{S_{\check{\alpha}}\cdot s^\lambda} = 0$. Same argument shows that σ in fact vanishes along any T-invariant curves in Gr_G^λ.

Assume that σ vanishes along all $(r-1)$-dimensional admissible T-invariant subvarieties in Gr_G^λ. We prove it also vanishes along all r-dimensional admissible T-invariant subvarieties. Let p be a closed point of Gr_G^λ contained in some r-dimensional admissible T-invariant subvariety. Without loss of generality, we could assume $p\in U_{\check{\alpha}_{i_1}}\cdots U_{\check{\alpha}_{i_r}}s^\lambda$, with $\check{\alpha}_{i_1},\dots,\check{\alpha}_{i_r}$ distinct, and $\check{\alpha}_{i_r}\in\Delta_\lambda$. Therefore, one could write $p = gp'$ with $g\in U_{\check{\alpha}_{i_1}}\cdots U_{\check{\alpha}_{i_{r-1}}}$ and $p' = \exp(ce_{\check{\alpha}_{i_r}})s^\lambda$ for some $c\in\mathbb{C}$. Let $C = gS_{\check{\alpha}_{i_r}}s^\lambda$ be the translation of $S_{\check{\alpha}_{i_r}}s^\lambda$ by g. Then C is a rational curve of degree one in Gr_G^λ, for $\mathcal{L}_G$ is $G_{\mathcal{O}}$-linearized. Observe that $p\in gS_{\check{\alpha}_{i_r}}s^\lambda$, and that gs^λ and $gs^{\lambda-\alpha_{i_r}}$ are also contained in this curve. Furthermore, they belong to $(r-1)$-dimensional admissible T-invariant subvarieties of Gr_G^λ. Since σ vanishes at gs^λ and $gs^{\lambda-\alpha_{i_r}}$ by induction, σ vanishes along C, and in particular, at p. $\square$

2.2.9. Theorem. *Theorem* 0.2.2 *holds for simple algebraic groups of type* A.

Proof. Every fundamental coweight is minuscule for simple algebraic groups of type A. $\square$

2.2.10. Proposition. *Assume* G *is a simple algebraic group of type* A, D, *or* E. *Then Theorem* 2.1.2 *holds if* λ *is the highest coroot.*

Proof. Recall that in the case G is simply-laced, the highest coroot is just θ. Let P be a minimal parabolic subgroup of G containing $U_{\check{\theta}}$ and $U_{-\check{\theta}}$. There is an obvious Levi subgroup of P whose derived group is $M\cong PSL_2$, and $\mathrm{Lie}\, M = \mathbb{C}\{e_{\check{\theta}},\theta,e_{-\check{\theta}}\}$. Let $\mathrm{Gr}_M\to\mathrm{Gr}_G$ be the embedding, induced from $M_{\mathcal{K}}\to\mathrm{Gr}_G$ by sending $m\mapsto ms^\theta$. Let $Z_\theta = \overline{M_{\mathcal{O}}s^\theta}\subset\mathrm{Gr}_M\cap\overline{\mathrm{Gr}}_G^\theta$. Then $Z_\theta\cong\overline{\mathrm{Gr}}_M^{p(\theta)}\cong\overline{\mathrm{Gr}}_{SL_2}^2$, where we identify the coweight lattice of $\mathfrak{sl}_2$ with $\mathbb{Z}$ by identifying the fundamental coweight with 1. This is T-invariant. Corollary 1.2.8 implies that $Z_\theta^T\cong(\overline{\mathrm{Gr}}_{SL_2}^2)^{T_{SL_2}}$. Furthermore, the restriction of $\mathcal{L}_G$ on $\overline{\mathrm{Gr}}_G^\theta$ to $\overline{\mathrm{Gr}}_{SL_2}^2$ is $\mathcal{L}_{SL_2}$ by Lemma 2.2.6, since $(\check{\theta},\check{\theta})^* = 2$. Now let $\sigma\in H^0(\overline{\mathrm{Gr}}_G^\theta,\mathcal{I}^\theta(1))$. Since Theorem 2.1.2 holds for SL_2, one obtains that $\sigma|_{Z_\theta} = 0$. Likewise, one can define Z_α for any α positive coroot, and for the same reason, $\sigma|_{Z_\alpha} = 0$.

Now observe that $\langle\theta,\check{\alpha}\rangle\leqslant 1$ for any $\check{\alpha}$ positive root other than $\check{\theta}$, while $\langle\theta,\check{\theta}\rangle = 2$. Denote $\Delta_\theta = \{\check{\alpha}\text{ positive}, \langle\check{\alpha},\theta\rangle = 1\}$. Then one can use the same method as in 2.2.8 to prove that σ indeed vanishes along any r-dimensional admissible T-invariant subvarieties in $\overline{\mathrm{Gr}}_G^\theta$. If s^α and s^β are connected by some T-invariant curve, then this curve is rational of degree 1 unless

$\alpha = -\beta$. For the former case, $\sigma = 0$ at s^α and s^β implies that σ vanishes along this curve. For the latter case, the rational curve is of degree 2, and is contained in Z_α. This proves the case $r = 1$, and $r > 1$ is proved by induction as in 2.2.8. Let p be a closed point of $\overline{\mathrm{Gr}}_G^\theta$ contained in some r-dimensional admissible T-invariant subvariety. Since $\overline{\mathrm{Gr}}_G^\theta - \mathrm{Gr}_G^\theta = s^0$, we could assume $p \in \mathrm{Gr}_G^\theta$. Observe that, as in 2.2.8, we could further assume that

$$p \in U_{\check{\psi}_1} U_{\check{\psi}_2} U_{\check{\psi}_3} \cdots U_{\check{\psi}_r} s^\theta$$

where $\check{\psi}_i$ are distinct and, $\check{\psi}_r \in \{\check{\theta}, \check{\delta} + \check{\theta}\} \cup \Delta_\theta$. If $\check{\psi}_r \neq \check{\theta}$, one writes $p = gp'$ with $g \in U_{\check{\psi}_1} \cdots U_{\check{\psi}_{r-1}}$ and $p' = \exp(ce_{\check{\psi}_r})s^\theta$ for some $c \in \mathbb{C}$. Since $\check{\psi}_r \neq \check{\theta}$, p is contained in the degree one rational curve $gS_{\check{\psi}_r} s^\theta$. (Observe that $S_{\check{\delta}+\check{\theta}} s^\theta$ is rational of degree one by Lemma 2.2.2.) Now argue as in 2.2.8 to show σ vanishes at p. Therefore, one could assume that $\check{\psi}_r = \check{\theta}$. Let $\check{\psi}_{r-1} = n_{r-1}\check{\delta} + \check{\beta}_{r-1}$. We could further assume that $\langle \check{\beta}_{r-1}, \theta \rangle < 0$. Otherwise, we can simply interchange $U_{\check{\psi}_{r-1}}$ and $U_{\check{\psi}_r}$ (since $[e_{\check{\theta}}, e_{n_{r-1}\check{\delta}+\check{\beta}_{r-1}}] = 0$ in this case), and return to the case that $\check{\psi}_r \neq \check{\theta}$. However, if $\langle \check{\beta}_{r-1}, \theta \rangle < 0$,

$$U_{n_{r-1}\check{\delta}+\check{\beta}_{r-1}} U_{\check{\theta}} s^\theta \subset U_{\check{\theta}} U_{n_{r-1}\check{\delta}+\check{\beta}_{r-1}+\check{\theta}} s^\theta.$$

We still return to the case that $\check{\psi}_r \neq \check{\theta}$. □

2.2.11. We need some more preparations in order to prove Theorem 2.1.2 for algebraic groups of type D. We temporarily do not assume that G is simply-laced. For any $\lambda \in \Lambda_G^+$, there is an embedding $G/P_\lambda \cong G \cdot s^\lambda \subset \overline{\mathrm{Gr}}_G^\lambda$, where P_λ is the parabolic subgroup of G generated by B^- and $U_{\check{\alpha}}$ with $\check{\alpha}$ positive and $\langle \check{\alpha}, \lambda \rangle = 0$. The point $s^{w\lambda} \in G/P_\lambda$ will simply be denoted by $w \in G/P_\lambda$.

It is not difficult to see that the restriction of $\mathcal{L}_G$ to G/P_λ is $\mathcal{O}(\iota\lambda) = G \times^{P_\lambda} \mathbb{C}_{\iota\lambda}$, where P_λ acts on $\mathbb{C}_{\iota\lambda}$ through the character $\iota\lambda$. Therefore $\Gamma(G/P_\lambda, \mathcal{L}_G) \cong V^{\iota\lambda}$ by the Borel–Weil theorem. We have the natural map $\Gamma(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G) \to \Gamma(G/P_\lambda, \mathcal{O}(\iota\lambda))$. Assume that λ is not minuscule. Let $Z = \overline{\mathrm{Gr}}_G^\lambda \setminus \mathrm{Gr}_G^\lambda$ endowed with the reduced scheme structure and $\mathcal{J}$ be the sheaf of ideals defining Z. It is clear that the composition

$$\Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{J} \otimes \mathcal{L}_G\big) \hookrightarrow \Gamma\big(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G\big) \twoheadrightarrow \Gamma\big(G/P_\lambda, \mathcal{O}(\iota\lambda)\big)$$

is nonzero (e.g. $\sigma_\lambda \neq 0$ at s^λ) and therefore surjective. Now assume that G is simply-laced. We have

2.2.12. Proposition. *Let $\lambda \in \Lambda_G^+$, which is not minuscule. If $\langle \lambda, \check{\alpha} \rangle \leqslant 2$ for any positive root $\check{\alpha}$, then $\Gamma(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{J} \otimes \mathcal{L}_G) \to \Gamma(G/P_\lambda, \mathcal{O}(\iota\lambda))$ is an isomorphism.*

Proof. Assume that $\sigma \in \Gamma(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G)$ with $\sigma|_{Z \cup G \cdot s^\lambda} = 0$. We prove that $\sigma = 0$. Recall that Gr_G^λ is an affine bundle over G/P_λ whose fiber at gs^λ ($g \in G$) is $g \prod_{\langle \check{\alpha}, \lambda \rangle = 2} U_{\check{\delta}+\check{\alpha}} s^\lambda$. Observe that each $gS_{\check{\delta}+\check{\alpha}} s^\lambda$ is a rational curve of degree one containing $gs^\lambda \in G/P_\lambda$ and $gs^{\lambda-\alpha} \in Z$, at which σ vanishes. Our method used in the proofs of Propositions 2.2.8 and 2.2.10 indicates that σ vanishes along the fiber over any $p \in G/P_\lambda$. □

2.2.13. Theorem. *Theorem* 2.1.2 *holds for simple algebraic groups of type D.*

Proof. Observe in this case, $\omega_1, \omega_{\ell-1}, \omega_\ell$ are minuscule, ω_2 is the highest coroot. Therefore, the theorem holds for D_4. Now assume that $\ell > 4$. We are aiming to prove that Theorem 2.1.2 holds for ω_i, $3 \leqslant i \leqslant \ell - 2$. Let

$$\check\beta_1 = \check\alpha_1, \quad \check\beta_2 = \check\alpha_2, \quad \ldots, \quad \check\beta_{i-1} = \check\alpha_{i-1}, \quad \check\beta_i = \check\alpha_{i-1} + 2\check\alpha_i + \cdots + 2\check\alpha_{\ell-2} + \check\alpha_{\ell-1} + \check\alpha_\ell.$$

Then $\check\beta_1, \ldots, \check\beta_i$ determines a subgroup of G, of type D_i, which we denote by M. One can show that M is the derived group of a Levi subgroup of G. Indeed, another set of simple roots $\{\check\alpha'_1, \ldots, \check\alpha'_\ell\}$ of G can be chosen as

$$-\check\alpha_{i+1}, \ldots, -\check\alpha_{\ell-1}, -\check\alpha_1 - \cdots - \check\alpha_{\ell-2} - \check\alpha_\ell, \check\beta_1, \ldots, \check\beta_i.$$

Then M is the derived group of a standard Levi factor for this set of simple roots. Let T_M be the maximal torus of M, whose Lie algebra is generated by $\beta_1, \ldots, \beta_i$. Denote $\Delta_i = \{\check\alpha, \langle \omega_i, \check\alpha\rangle = 2\}$. Observe that

$$\prod_{\check\alpha \in \Delta_i} U_{\check\alpha} U_{\check\delta + \check\alpha} s^{\omega_i} = M_{\mathcal{O}} s^{\omega_i}.$$

We denote its closure by Z_{ω_i}. Then $Z_{\omega_i} \cong \overline{\mathrm{Gr}}_M^{2\omega_i^M}$, where ω_i^M is the fundamental coweight of M corresponding to $\check\beta_i$. We have: (i) the pullback of $\mathcal{L}_G$ to Z_{ω_i} is isomorphic to $\mathcal{L}_M$ by Lemma 2.2.6; (ii) $(Z_{\omega_i})^T \cong (\overline{\mathrm{Gr}}_M^{2\omega_i^M})^{T_M}$ by Corollary 1.3.8.

Let $\sigma \in H^0(\overline{\mathrm{Gr}}_G^{\omega_i}, \mathcal{I}^{\omega_i}(1))$, then by induction for ℓ, $\sigma|_{Z_{\omega_i}} = 0$. Likewise, for any $w\omega_i$, one can define $Z_{w\omega_i}$ and prove similarly that $\sigma|_{Z_{w\omega_i}} = 0$. The goal is to show that $\sigma = 0$ and therefore $H^0(\overline{\mathrm{Gr}}_G^{\omega_i}, \mathcal{I}^{\omega_i}(1)) = 0$. Since $H^0(\overline{\mathrm{Gr}}_G^{\omega_i}, \mathcal{I}^{\omega_i}(1))$ is a T-module, we could assume that σ is a T-weight vector.

Recall the settings as in 2.2.11. In this case we have $\mathcal{J}$ the ideal defining $\overline{\mathrm{Gr}}_G^{\omega_{i-2}}$ in $\overline{\mathrm{Gr}}_G^{\omega_i}$ and $\Gamma(\overline{\mathrm{Gr}}_G^{\omega_i}, \mathcal{J} \otimes \mathcal{L}_G) \cong \Gamma(G/P_i, \mathcal{O}(\check\omega_i))$, where $P_i = P_{\omega_i}$ is the parabolic subgroup generated by B_- and $U_{\check\alpha}$ with $\langle \omega_i, \check\alpha\rangle = 0$. By induction on i, we could assume that $\sigma \in \Gamma(\overline{\mathrm{Gr}}_G^{\omega_i}, \mathcal{J} \otimes \mathcal{L}_G)$. Then our theorem is a consequence of the following proposition. $\square$

2.2.14. Proposition. *Let $\sigma \in \Gamma(G/P_i, \mathcal{O}(\check\omega_i))$. If σ is a T-weight vector (or equivalently,* $\mathrm{supp}(\sigma)$ *is T-invariant), and vanishes along any*

$$Z_w := \Bigl(\prod_{\langle w\omega_i, \check\alpha\rangle = 2} U_{\check\alpha}\Bigr) w, \quad w \in W,$$

then $\sigma = 0$.

Proof. Recall that $\Gamma(G/P_{\omega_i}, \mathcal{L}_G) = V^{\check\omega_i}$. The anti-dominant weights appearing in $V^{\check\omega_i}$ are $\check\omega_i, \check\omega_{i-2}, \ldots$. Thus, without loss of generality, we could assume that $\sigma \in V^{\check\omega_i}(\check\omega_{i-2k})$ where $2k \leqslant i$. We will deduce the proposition from the following lemma, whose proof is left to the readers. For any $w \in W$, the $(-w\check{\omega}_i)$-weight space $V^{\check{\omega}_i}(-w\check{\omega}_i)$ is one-dimensional. Pick up an extremal vector $0 \neq v_w \in V^{\check{\omega}_i}(-w\check{\omega}_i)$ for each w.

2.2.15. Lemma. *$V^{\check{\omega}_i}(-\check{\omega}_{i-2k})$ has a basis of the forms*

$$e_{\check{\beta}_{d,1}} e_{\check{\beta}_{d,2}} \cdots e_{\check{\beta}_{d,k}} v_{w_d}, \quad w_d \in W,\ d = 1, 2, \ldots, \dim V^{\check{\omega}_i}(-\check{\omega}_{i-2k}),$$

with $(\check{\beta}_{d,j}, w_d\check{\omega}_i)^* = 2$ and $(\check{\beta}_{d,j}, \check{\beta}_{d,j'})^* = 0$ for $j \neq j'$.

Observe that $(V^{\check{\omega}_i})^* \cong V^{\check{\omega}_i}$ via the inner product. Choose a basis of $V^{\check{\omega}_i}(-\check{\omega}_{i-2k})$ of the forms $u_d = e_{\check{\beta}_{d,1}} e_{\check{\beta}_{d,2}} \cdots e_{\check{\beta}_{d,k}} v_{w_d}$ as in the lemma. Let $\{u_d^*\}$ be the basis of $V^{\check{\omega}_i}(-\check{\omega}_{i-2k})^* = V^{\check{\omega}_i}(\check{\omega}_{i-2k})$ dual to $\{u_d\}$ and write $\sigma = \sum_d \sigma_d u_d^*$ where $\sigma_d \in \mathbb{C}$. Regard $u_d^* \in \Gamma(G/P_i, \mathcal{O}(\check{\omega}_i))$ by embedding G/P_i into $\mathbb{P}(V^{\check{\omega}_i})$ through $g \mapsto gv_1$, then u_d^* does not vanish along the whole $U_{\check{\beta}_{d,1}} U_{\check{\beta}_{d,2}} \cdots U_{\check{\beta}_{d,k}} w_d$, while $u_{d'}^*$ does vanish along it for any $d' \neq d$. Since σ vanishes on $U_{\check{\beta}_{d,1}} U_{\check{\beta}_{d,2}} \cdots U_{\check{\beta}_{d,k}} w_d$ for all d, $\sigma = 0$. □

2.2.16. The main Theorem 0.2.2 now is proved for G of type A or D. Let us also discuss some cases that we can prove now for simple algebraic groups of type E.

For G being of type E_6, ω_1 and ω_6 are minuscule, and ω_2 is the highest coroot; for G being of type E_7, ω_1 is the highest coroot, ω_7 is minuscule; for G being of type E_8, ω_8 is the highest coroot. We could also prove

2.2.17. Proposition. *Theorem* 2.1.2 *also holds in the following cases:* (i) *G is of type E_6, and $\lambda = \omega_3$ or ω_5;* (ii) *G is of type E_7, and $\lambda = \omega_2$ or ω_6;* (iii) *G is of type E_8, and $\lambda = \omega_1$.*

Proof. The proofs are similar to the previous ones. Take $G = E_6$ and $\lambda = \omega_3$ for example. Let

$$\check{\beta}_1 = \check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\alpha}_4 + \check{\alpha}_6, \quad \check{\beta}_2 = \check{\alpha}_5, \quad \check{\beta}_3 = \check{\alpha}_4, \quad \check{\beta}_4 = \check{\alpha}_3, \quad \check{\beta}_5 = \check{\alpha}_6.$$

Then $\{\check{\beta}_1, \ldots, \check{\beta}_5\}$ determines a subgroup M of G, of type A_5, which is the derived group of a Levi subgroup of G. (To see this, observe that $\{\check{\beta}_1, \check{\beta}_2, \check{\beta}_3, \check{\beta}_4, \check{\beta}_5, -\check{\alpha}_2 - 2\check{\alpha}_3 - 2\check{\alpha}_4 - \check{\alpha}_5 - \check{\alpha}_6\}$ can be chosen as the set of simple roots for G.) Denote $\Delta_3 = \{\check{\alpha}, \langle \omega_3, \check{\alpha} \rangle = 2\}$. Then the closure of $\prod_{\check{\alpha} \in \Delta_3} U_{\check{\alpha}} U_{\check{\delta}+\check{\alpha}} s^{\omega_3}$ in $\overline{\mathrm{Gr}}_G^{\omega_2}$, is isomorphic to $\overline{\mathrm{Gr}}_M^{2\omega_1^M}$, where ω_1^M is the fundamental coweight of M corresponding to $\check{\beta}_1$. Now proceed as in 2.2.13. □

2.3. Applications: the smooth locus of $\overline{\mathrm{Gr}}_G^{\lambda}$

We prove Corollary 0.2.4 in this section. It is clear that to prove the corollary, it is enough to prove that $\overline{\mathrm{Gr}}_G^{\lambda}$ is not smooth at s^{μ} for $\mu \in \Lambda_G^+$, $\mu < \lambda$. Since $(\overline{\mathrm{Gr}}_G^{\lambda})^T$ is a finite scheme, a direct consequence of Theorem 0.2.2 is the following

2.3.1. Corollary. *In the notations of Theorem* 0.2.2*, let $V_\lambda = H^0(\overline{\mathrm{Gr}}_G^{\lambda}, \mathcal{L}_G)^*$ be the affine Demazure module. Then*

$$\dim V_\lambda(-\iota\mu) = \mathrm{length}_{\mathbb{C}} \mathcal{O}_{(\overline{\mathrm{Gr}}_G^{\lambda})^T, s^{\mu}}$$

where $V_\lambda(-\iota\mu)$ denotes the $(-\iota\mu)$-weight subspace of V_λ.

Proof. Observe that under the isomorphism in Theorem 0.2.2, $\mathcal{O}_{(\overline{\mathrm{Gr}}_G^\lambda)^T, s^\mu}$ corresponds to the $(\iota\mu)$-weight subspace in $\Gamma(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G)$. $\square$

To prove that $\overline{\mathrm{Gr}}_G^\lambda$ is not smooth at s^μ, we will need the following simple and well-known lemmas.

2.3.2. Lemma. *Let $X = \operatorname{Spec} A$ be an affine scheme over k with an action of a torus T. Then A is an algebraic representation of T. Decompose $A = A_0 \oplus A_+$, where A_0 is the zero weight space and A_+ is the direct sum of nonzero weight spaces. Let X^T be the T-fixed subcheme of X defined as above and I the ideal defining X^T. Then $I = A_+A$.*

2.3.3. Lemma. *Let X be a smooth quasi-projective variety with an action of a torus T. Then X^T is also smooth, and in particular, is reduced.*

Proof. Since X is covered by T-invariant open affine subschemes, we could assume that X is affine. Let $x \in X^T$, then $\operatorname{Spec} \mathcal{O}_{X,x}$ is a T-invariant subscheme of X. Therefore, we could replace our X by $\operatorname{Spec} A$, where $A = \mathcal{O}_{X,x}$. Let $\mathfrak{m}$ be the maximal ideal of A. Then one can find a T-invariant subspace N in $\mathfrak{m}$, such that the natural projection $N \to \mathfrak{m}/\mathfrak{m}^2$ is a T-module isomorphism. Decompose $N = N_0 \oplus N_+$, where N_0 is the zero weight space and N_+ is the direct sum of some nonzero weight spaces. Since A is regular, N generate $\mathfrak{m}$. By the lemma above, it is easy to see that $X^T = \operatorname{Spec}(A/N_+)$, which is obviously regular. $\square$

Now, according to Corollary 2.3.1 and Lemma 2.3.3, to prove Corollary 0.2.4, it is enough to show that

2.3.4. Lemma. *For a simple algebraic group G of type A or D, let $\lambda, \mu \in \Lambda_G^+$, $\mu \leqslant \lambda$, then* $\dim V_\lambda(-\iota\mu) > 1$.

Proof. We need the following

2.3.5. Sublemma. *For any simple, not necessarily simply-laced algebraic group G, the natural morphism $H^0(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G^{\otimes k}) \to H^0(\overline{\mathrm{Gr}}_G^\mu, \mathcal{L}_G^{\otimes k})$ is surjective.*

Assuming the above sublemma, we first conclude the proof of the proposition, and in consequence, Corollary 0.2.4. Recall that $\tilde{G}$ denotes the simply-connected cover of G. The sublemma implies that as $\tilde{G}$-modules, V_μ is a direct summand of V_λ. However, V_λ is a $\tilde{G}$-module of lowest weight $-\iota\lambda$. Therefore, V_λ contains the simple $\tilde{G}$-module $V^{-w_0\iota\lambda}$ of highest weight $-w_0\iota\lambda$ as a direct summand, where w_0 is the element in the Weyl group of maximal length. Since V_μ is of lowest weight $-\iota\mu$, $V_\mu \cap V^{-w_0\iota\lambda} = \emptyset$ in V_λ. Observe that $\iota\mu \in \check{\Lambda}_+$ and $\iota\mu \leqslant \iota\lambda$. Whence, $\dim V^{-w_0\iota\lambda}(-\iota\mu) \geqslant 1$. This together with $\dim V_\mu(-\iota\mu) = 1$ implies the lemma.

It remains to prove the sublemma. When G is of type A or D and $k = 1$, which is the case we need, the sublemma is easily proved as follows. Apply the main theorem. Then the surjectivity of $H^0(\overline{\mathrm{Gr}}_G^\lambda, \mathcal{L}_G) \to H^0(\overline{\mathrm{Gr}}_G^\mu, \mathcal{L}_G)$ is equivalent to the surjectivity of $\mathcal{O}_{(\mathrm{Gr}_G^\lambda)^T} \to \mathcal{O}_{(\mathrm{Gr}_G^\mu)^T}$, which is obvious.

However, let us include a proof of the full statement of the sublemma for completeness. This is well known and in fact is the one of the ingredients for the proof of the Demazure character formula in [24] and [27]. (Therefore, the previous easy proof for G simply-laced and $k=1$ does not quite apply, since we used the main theorem, whose proof relies on [24] and [27].) The statement of the sublemma is the direct consequence of the following two facts. First, there exists a flat model $\overline{\mathrm{Gr}}^{\lambda}_G$ over $\mathbb{Z}$ such that: (i) if $\mu \leqslant \lambda$, $\overline{\mathrm{Gr}}^{\mu}_G$ is a closed subscheme of $\overline{\mathrm{Gr}}^{\lambda}_G$; (ii) over an open subset of $\operatorname{Spec}\mathbb{Z}$, the geometrical fibers are reduced and therefore are the Schubert varieties over that base field (e.g. see [28, Section 3, Lemma 3]). Second, over an algebraically closed field of characteristic p, $\overline{\mathrm{Gr}}^{\mu}_G \subset \overline{\mathrm{Gr}}^{\lambda}_G$ are compatibly Frobenius splitting. This is in fact proved in [11] and [27] for Schubert varieties in the affine flag variety $\mathcal{F}\ell_G := G_{\mathcal{K}}/I$, where I is the Iwahori subgroup. However, it is known that for any $\overline{\mathrm{Gr}}^{\lambda}_G$, there exist Schubert varieties $X_{w_\mu} \subset X_{w_\lambda}$ in $\mathcal{F}\ell_G$ such that the projection $X_{w_\lambda} \to \overline{\mathrm{Gr}}^{\lambda}_G$ is proper birational, and under the projection, the scheme theoretical image of X_{w_μ} is $\overline{\mathrm{Gr}}^{\mu}_G$. The normality of $\overline{\mathrm{Gr}}^{\lambda}_G$ (cf. [11, Theorem 8]) together with [28, Section 1, Proposition 4] implies that $\overline{\mathrm{Gr}}^{\mu}_G \subset \overline{\mathrm{Gr}}^{\lambda}_G$ are compatibly Frobenius splitting. Then by [28, Section 1, Proposition 3], $H^0(\overline{\mathrm{Gr}}^{\lambda}_G, \mathcal{L}_G^{\otimes k}) \to H^0(\overline{\mathrm{Gr}}^{\mu}_G, \mathcal{L}_G^{\otimes k})$ is surjective over $\bar{\mathbb{F}}_p$. Now the first statement and the semi-continuity theorem [17, Theorem 7.7.5] imply that the sublemma also holds in characteristic 0. □

3. The bosonic realization

We will discuss the geometrical form of the FKS in this section. The language we will be using is the factorization algebras developed by Beilinson and Drinfeld in [5, Section 3.4] (see also [13, Chapter 20]). We refer the readers to [13, Chapters 19 and 20] for the dictionaries between vertex algebras and factorization algebras. We will fix a smooth curve X throughout this section. The canonical sheaf ω_{X^n} on X^n is regarded as a right D-module. For any morphism $f: M \to N$, $f_!$ denotes the push-forward of right D-modules.

3.1. The Heisenberg algebras

3.1.1. Let $\mathfrak{t}$ be an abelian Lie algebra, and $(\cdot,\cdot)$ be a symmetric bilinear form on $\mathfrak{t}$. Given such data, we define the Heisenberg Lie algebra to be the central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{t}} \to \mathfrak{t} \hat{\otimes} \mathcal{K} \to 0$$

with the Lie bracket given by

$$[A \otimes f, B \otimes g] = (A, B)\operatorname{Res}(g\,df)K \quad \text{for } A, B \in \mathfrak{t},\ f, g \in \mathcal{K}.$$

For any $\lambda \in \mathfrak{t}$, the level k Fock module π^k_λ of $\hat{\mathfrak{t}}$ is defined as $\pi_\lambda = \operatorname{Ind}^{\hat{\mathfrak{t}}}_{\mathfrak{t}\hat{\otimes}\mathcal{O}\oplus\mathbb{C}K}\mathbb{C}$, where K acts on $\mathbb{C}$ by multiplication by k, $\mathfrak{t} \hat{\otimes} \mathfrak{m}$ acts on $\mathbb{C}$ by zero ($\mathfrak{m}$ the maximal ideal of $\mathcal{O}$), and $\mathfrak{t} = \mathfrak{t} \hat{\otimes} \mathcal{O}/\mathfrak{t} \hat{\otimes} \mathfrak{m}$ acts on $\mathbb{C}$ by $\iota\lambda$, $\iota: \mathfrak{t} \to \mathfrak{t}^*$ being the isomorphism induced from the non-degenerate bilinear form on $\mathfrak{t}$. Level one Fock modules π^1_λ are usually simply denoted by π_λ. It is clear that the Fock modules are irreducible $\hat{\mathfrak{t}}$-modules if the bilinear form $(\cdot,\cdot)$ is non-degenerate.

Given a simple algebraic group G, we obtain the data $(\mathfrak{t}, (\cdot,\cdot))$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, and $(\cdot,\cdot)$ is the restriction to $\mathfrak{t}$ of the normalized invariant form on $\mathfrak{g}$. Then the Heisenberg algebra $\hat{\mathfrak{t}}$ associated to this data is just the restriction of the central extension of $\mathfrak{g}\hat{\otimes}\mathcal{K}$ to $\mathfrak{t}\hat{\otimes}\mathcal{K}$. Therefore, any module over $\hat{\mathfrak{g}}$ is a module over $\hat{\mathfrak{t}}$.

All the discussions in this section are based on the following theorem.

3.1.2. Proposition. *Let G be a simple (not necessarily simply-connected) algebraic group of type A, D, or E. For any $\gamma \in \pi_1(G)$, recall ω_{i_γ} from* 1.4.3. *Then one has the isomorphism*

$$L(\Lambda + \iota\omega_{i_\gamma}) \cong \bigoplus_{\lambda \in R_G} \pi_{\omega_{i_\gamma}+\lambda}$$

as $\hat{\mathfrak{t}}$-modules.

Proof. By Proposition 1.4.4 and Theorem 1.3.4, it suffices to prove that the natural morphism

$$\Gamma\big((\mathrm{Gr}_G)^\gamma, \mathcal{L}_G\big) \to \Gamma\big((\mathrm{Gr}_G)^\gamma, \mathcal{O}_{((\mathrm{Gr}_G)^\gamma)^T} \otimes \mathcal{L}_G\big)$$

is an isomorphism. If G is of type A or D, this directly follows from Theorem 0.2.2 since $(\mathrm{Gr}_G)^\gamma = \varinjlim \overline{\mathrm{Gr}}^\lambda_G$, where limit is taken over $(\omega_{i_\gamma} + \Lambda_G^+)$. To give a proof of the proposition also applicable to type E, we make use of the following simple observation.

Claim. *For any λ a dominant coweight, there exists n big enough such that $\lambda \leqslant n\theta$.*

Therefore, $(\mathrm{Gr}_G)^\gamma = \varinjlim \overline{\mathrm{Gr}}^\lambda_G$, where the limit now is taken over $(\omega_{i_\gamma} + n\theta)$, and the proposition follows from Propositions 2.1.1, 2.1.3, 2.2.8 and 2.2.10. □

3.2. Lattice factorization algebras vs. affine Kac–Moody factorization algebras

3.2.1. Let L be a lattice with a symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$ such that $(\lambda,\lambda) > 0$ for all $\lambda \in L \setminus \{0\}$. For simplicity, we will always assume that L is an even lattice, i.e. $(\lambda,\lambda) \in 2\mathbb{Z}$ for any $\lambda \in L$. Set $T = L \otimes_{\mathbb{Z}} \mathbb{G}_m$. This is a torus with Lie algebra $\mathfrak{t} = L \otimes \mathbb{C}$ and coweight lattice $L \subset \mathfrak{t}$. We still denote $(\cdot,\cdot)$ the bilinear form on $\mathfrak{t}$ obtained by the extension of the one on L. Let $\hat{\mathfrak{t}}$ be the Heisenberg algebra corresponding to $(\mathfrak{t}, (\cdot,\cdot))$.

Choose a 2-cocycle $\varepsilon : L \times L \to \mathbb{Z}/2$ such that $\varepsilon(\lambda,\mu) + \varepsilon(\mu,\lambda) = (\lambda,\mu) \bmod 2$. Attached to this data, there is a canonical symmetric central extension $\tilde{T}_K$ of $T_{\mathcal{K}}$ by $\mathbb{G}_m$, with a canonical splitting $i : T_{\mathcal{O}} \to \tilde{T}_K$ (cf. [4] and [15, Section 6.1.1]), such that the commutator pairing $T_{\mathcal{K}} \times T_{\mathcal{K}} \to \mathbb{G}_m$ is given by

$$\big\{\lambda(f), \mu(g)\big\} = \{f, g\}^{-(\lambda,\mu)}. \tag{6}$$

Here we regard $\lambda \in L$ as morphism $(\mathbb{G}_m)_{\mathcal{K}} \to T_{\mathcal{K}}$ and $\{\cdot,\cdot\} : (\mathbb{G}_m)_{\mathcal{K}} \times (\mathbb{G}_m)_{\mathcal{K}} \to \mathbb{G}_m$ is the Contou–Carrère symbol. This central extension defines an invertible sheaf $\mathcal{L}_T$ over $\mathrm{Gr}_T = T_{\mathcal{K}}/T_{\mathcal{O}}$. Observe that in our convention, $\mathcal{L}_T$ corresponds to $(R_Q)^{-1}$ as in [15, Section 6.1.1].

Recall that as topological spaces, $(\mathrm{Gr}_T)_{\mathrm{red}} = \{s^\lambda\}$ for $\lambda \in L$. Denote δ_λ the right D-module of delta-functions at s^λ, i.e. $\delta_\lambda = (s^\lambda)_!\mathbb{C}$, where s^λ is regarded as the map $\operatorname{Spec}\mathbb{C} \to \mathrm{Gr}_T$. Define

$$V_L = \Gamma\left(\mathrm{Gr}_T, \left(\bigoplus_{\lambda \in L} \delta_\lambda\right) \otimes \mathcal{L}_T^{-1}\right) \cong \Gamma(\mathrm{Gr}_T, \mathcal{L}_T)^*.$$

Then it is clear that as $\hat{\mathfrak{t}}$-modules

$$V_L \cong \bigoplus_{\lambda \in L} \pi_\lambda.$$

3.2.2. If we begin with the global curve X, then the connected components of Gr_{T,X^n} are labelled by $(\lambda_1, \dots, \lambda_n) \in L^n$, and the reduced part of each connected component is isomorphic to X^n. Let $s^{\lambda_1,\dots,\lambda_n}$ be the corresponding section $X^n \to \mathrm{Gr}_{T,X^n}$ and $\delta_{\lambda_1,\dots,\lambda_n}$ the corresponding sheaf of delta-functions. I.e. $\delta_{\lambda_1,\dots,\lambda_n} = (s^{\lambda_1,\dots,\lambda_n})_! \omega_{X^n}$.

There are canonical line bundles $\mathcal{L}_{T,X^n}$ over Gr_{T,X^n}, which are obtained from $\mathcal{L}_T$ by moving points. They satisfy the factorization property since so does the Contou–Carrère symbol (cf. [4, 2.3]). There is another description of $\{\mathcal{L}_{T,X^n}\}$ when X is complete, similar to the ones for a simple algebraic group as in 1.1.9. Namely, there is a canonical line bundle $\mathcal{L}_T$ on $Bun_{T,X}$, and $\mathcal{L}_{T,X^n}$ are the pullbacks of $\mathcal{L}_T$ by $\pi_n : \mathrm{Gr}_{T,X^n} \to Bun_{T,X}$. We will simply denote them by $\mathcal{L}_T$ in the following. Let $p_n : \mathrm{Gr}_{T,X^n} \to X^n$ be the natural projections. Then

$$\left\{ \mathcal{G}_n := (p_n)_* \left(\bigoplus_{(\lambda_1,\dots,\lambda_n) \in L^n} \delta_{\lambda_1,\dots,\lambda_n} \otimes \mathcal{L}_T^{-1} \right) \otimes \omega_{X^n}^{-1} \cong \big((p_n)_* \mathcal{L}_T \big)^* \right\}$$

form a factorization algebra (cf. [4, Proposition 3.3]). The vertex algebra structure on the fiber $\mathcal{G}_1 \otimes \mathbb{C}_x \cong V_L$ is called the lattice vertex algebra. The embedding

$$(p_n)_* \big(\delta_{0,\dots,0} \otimes \mathcal{L}_T^{-1} \big) \otimes \omega_{X^n}^{-1} \to \mathcal{G}_n$$

identifies the Heisenberg factorization algebra as a subalgebra of the lattice factorization algebra.

3.2.3. Let G be the simple simply-connected algebraic group whose Lie algebra is $\mathfrak{g}$. Recall the Beilinson–Drinfeld Grassmannian Gr_{G,X^n} from 1.1.6 and the invertible sheaf $\mathcal{L}_G$ from 1.1.9. For any n, *the* trivial G-bundle gives a section $e_n : X^n \to \mathrm{Gr}_{G,X^n}$. In other words, $e_2 = s^{0,0}$ as defined in 1.1.7. On the other hand, one has the projection $p_n : \mathrm{Gr}_{G,X^n} \to X^n$. Then the collection

$$\left\{ \mathcal{V}_n := (p_n)_* \big(\mathcal{L}_G^{\otimes(-k)} \otimes (e_n)_! \omega_{X^n} \big) \otimes \omega_{X^n}^{-1} \right\}$$

form a factorization algebra. For any $x \in X$ a closed point,

$$\mathcal{V}_1 \otimes \mathbb{C}_x \cong \Gamma \big(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes(-k)} \otimes \mathrm{IC}_0 \big) \cong \mathbb{V}(k\Lambda)$$

as $\hat{\mathfrak{g}}$-modules. Here IC_λ is the irreducible D-module on Gr_G whose support is $\overline{\mathrm{Gr}}_G^\lambda$, and

$$\mathbb{V}(k\Lambda) = \mathrm{Ind}_{\mathfrak{g} \hat{\otimes} \mathcal{O} + \mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}$$

is the level k vacuum module, on which $\mathfrak{g} \hat{\otimes} \mathcal{O}$ acts through the trivial character and K acts by multiplication by k. The factorization structure endows $\mathcal{V}_1 \otimes \mathbb{C}_x \cong \mathbb{V}(k\Lambda)$ with a vertex algebra structure, which is isomorphic to the standard affine Kac–Moody vertex algebra (cf. [13, Proposition 20.4.3] and [15, Theorem 5.3.1]).

3.2.4. Let $\{\mathcal{F}_n := ((p_n)_*\mathcal{L}_G^{\otimes k})^*\}$. Then each $\mathcal{F}_n$ is a quasi-coherent sheaf on X^n. (However, $(p_n)_*\mathcal{L}_G^{\otimes k}$ is not quasi-coherent.) It is clear that for any $x \in X$ closed point, $\mathcal{F}_1 \otimes \mathbb{C}_x \cong (\Gamma(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k}))^* \cong L(k\Lambda)$ as $\hat{\mathfrak{g}}$-modules. Moreover, the factorization property of $\mathcal{L}_G$ implies that

3.2.5. Lemma. *$\{\mathcal{F}_n\}$ form a factorization algebra. Furthermore, there is a natural morphism of factorization algebras $\{\mathcal{V}_n\} \to \{\mathcal{F}_n\}$ commuting with the fiberwise $\hat{\mathfrak{g}}$-action.*

Proof. It is clear that the factorization structure of $\{\mathcal{F}_n\}$ comes from the factorization property of $\mathcal{L}_G$ as indicated in 1.1.9. The unit is given by $(p_1)_*\mathcal{L}_G^{\otimes k} \to (p_1)_*(\mathcal{L}_G^{\otimes k} \otimes \mathcal{O}_{\overline{\mathrm{Gr}}^0_{G,X}}) \cong \mathcal{O}_X$. Here, $\overline{\mathrm{Gr}}^\lambda_{G,X}$ is the global analogue of $\overline{\mathrm{Gr}}^\lambda_G$ (see 1.1.7).

To construct a morphism $\{\mathcal{V}_n\} \to \{\mathcal{F}_n\}$ between factorization algebras which commutes with the $\hat{\mathfrak{g}}$-action, we first construct an $\mathcal{O}_{X^n}$-linear map

$$(p_n)_*\big((e_n)_!\omega_{X^n}\big) \to \omega_{X^n}.$$

We need

Claim. *If $f : M \to N$ is a smooth morphism of smooth algebraic varieties. Then for any $\mathcal{F}$ a right D-module on M, there is a natural morphism $Rf_*\mathcal{F} \to f_!\mathcal{F}$ in the derived category of $\mathcal{O}_N$-modules, where $f_!$ is the push-forward of D-modules.*

For any smooth algebraic variety M, D_M denotes the sheaf of differential operators on M. For $f : M \to N$ a morphism between smooth algebraic varieties, $D_{M\to N}$ denotes the $(D_M \times f^{-1}D_N)$-bimodule on M, whose underlying $\mathcal{O}_M$-module structure is isomorphic to f^*D_N. For $\mathcal{F}$ a right D-module on M, the push-forward $f_!\mathcal{F}$ to N is

$$f_!\mathcal{F} = Rf_*\big(\mathcal{F} \otimes^L_{D_M} D_{M\to N}\big).$$

If $f : M \to N$ is a smooth morphism, $D_{M\to N}$ has a resolution as left D_M-modules by $D_M \otimes \bigwedge^{-\cdot} \Theta_{M/N}$, where $\Theta_{M/N}$ is the relative tangent sheaf. Therefore, one obtains the natural morphism $\mathcal{F} \to \mathcal{F} \otimes \bigwedge^{-\cdot} \Theta_{M/N} \cong \mathcal{F} \otimes^L_{D_M} D_{M\to N}$. The claim is proved.

Now, since $p_n e_n = id$, by the claim, one obtains

$$(p_n)_*\big((e_n)_!\omega_{X^n}\big) \to (p_n)_!\big((e_n)_!\omega_{X^n}\big) \cong \omega_{X^n}.$$

Together with the natural $\mathcal{O}_{X^n}$-module morphism

$$(p_n)_*\big(\mathcal{L}_G^{\otimes(-k)} \otimes (e_n)_!\omega_{X^n}\big) \otimes \omega_{X^n}^{-1} \otimes (p_n)_*\mathcal{L}_G^{\otimes k} \to (p_n)_*\big((e_n)_!\omega_{X^n}\big) \otimes \omega_{X^n}^{-1}$$

one obtains the map

$$(p_n)_*\big(\mathcal{L}_G^{\otimes(-k)} \otimes (e_n)_!\omega_{X^n}\big) \otimes \omega_{X^n}^{-1} \otimes (p_n)_*\mathcal{L}_G^{\otimes k} \to \mathcal{O}_{X^n}.$$

In this way, one constructs a map $\{\mathcal{V}_n\} \to \{\mathcal{F}_n\}$. It is easy to check that it satisfies all the required properties. $\square$

3.2.6. Remark. (i) The factorization structure of $\{\mathcal{F}_n\}$ endows $L(k\Lambda) \cong \mathcal{F}_1 \otimes \mathbb{C}_x$ with a vertex algebra structure. By the lemma above, it is realized as a quotient of the affine Kac–Moody vertex algebra $\mathbb{V}(k\Lambda)$, and therefore, coincides with the usual vertex algebra structure on the integrable representation of level k.

(ii) Since $L(k\Lambda + \check{\nu})$ is a smooth $\hat{\mathfrak{g}}$-module, it is a module over the vertex algebra $\mathbb{V}(k\Lambda)$ (cf. [13, Theorem 5.1.6]). However, it is known that the action of $\mathbb{V}(k\Lambda)$ on $L(k\Lambda + \check{\nu})$ factors through $L(k\Lambda)$. Therefore, $L(k\Lambda + \check{\nu})$ is a module over $L(k\Lambda)$.

3.3. The Frenkel–Kac–Segal isomorphism

Let G be a simple, simply-connected algebraic group of A, D, E type, with Lie algebra $\mathfrak{g}$. Then the coroot lattice R_G (= coweight lattice in this case) together with the normalized invariant form on $\mathfrak{g}$ is an even lattice. The torus $T = R_G \otimes_{\mathbb{Z}} \mathbb{G}_m$ is the maximal torus of G. We have

3.3.1. Lemma.

(i) *The restriction of $\mathcal{L}_G$ to $\mathrm{Gr}_T \subset \mathrm{Gr}_G$ is just $\mathcal{L}_T$ associated to the invariant form, i.e. $\mathcal{L}_T \cong \mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_T}$.*

(ii) *The restrictions of $\mathcal{L}_G$ on Gr_{G,X^n} to $\mathrm{Gr}_{T,X^n} \subset \mathrm{Gr}_{G,X^n}$ are canonically isomorphic to $\mathcal{L}_T$ on Gr_{T,X^n}. Furthermore, such restrictions are compatible with all the factorizations.*

Proof. We only give a rough explanation since we did not give the precise definitions of $\mathcal{L}_T$ and $\mathcal{L}_G$. Indeed, $\mathcal{L}_G$ on Gr_G (respectively $\mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_T}$ on Gr_T) corresponds to the inverse of the $\mathbb{G}_m$-central extension of $G_{\mathcal{K}}$ (respectively $T_{\mathcal{K}}$) obtained by pullback the central extension

$$1 \to \mathbb{G}_m \to \mathbf{GL}\big(L_1(\Lambda)\big) \to \mathbf{PGL}\big(L_1(\Lambda)\big) \to 1.$$

We call it the determinantal central extension. It is proved in [22] (for the case $G = SL_2$) that the commutator pairing of this central extension is given by formula (6). This proves (i).

By (i), the pullback of the canonical line bundle $\mathcal{L}_G$ on $Bun_{G,X}$ (see 1.1.9) to $Bun_{T,X}$ is isomorphic to the canonical line bundle $\mathcal{L}_T$ on $Bun_{T,X}$ as introduced in 3.2.2. Then (ii) is clear. □

Putting things together, the geometrical form of the FKS isomorphism is

3.3.2. Theorem. *The dual of the natural morphism $(\pi_n)_*\mathcal{L}_G \to (\pi_n)_*(\mathcal{L}_G \otimes \mathcal{O}_{\mathrm{Gr}_{T,X^n}})$ gives an isomorphism of factorization algebras $\{\mathcal{G}_n\} \to \{\mathcal{F}_n\}$.*

Proof. By the lemma above, this is a factorization algebra morphism. It remains to prove that this is an isomorphism, which can be checked fiberwise, and is proved in Theorem 3.1.2. □

3.3.3. Finally, let us explain why Theorem 0.2.6 is true. Pick up a point $x \in X$. Then the category of $\{\mathcal{G}_n\}$-modules supported at x is semisimple, with all simple objects labelled by $\gamma \in L'/L$, where $L' = \{\lambda \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\lambda, \mu) \in \mathbb{Z}, \forall \mu \in L\}$ is the dual lattice (cf. [9] and [4, Lemma 1.9 and Proposition 3.8]). For each $\gamma \in L'/L$, the corresponding module is

$$V_L^{\gamma} = \bigoplus_{\lambda \in \gamma + L} \pi_{\lambda}.$$

Now let $L = R_G$ be the coroot lattice of G. Since $V^{\gamma}_{R_G}$ is a simple $\{\mathcal{G}_n\}$-module, it is a simple $\{\mathcal{F}_n\}$-module, and therefore is a simple $\hat{\mathfrak{g}}$-module. According to Theorem 3.1.2, $V^{\gamma}_{R_G}$ is isomorphic to $L(\Lambda + \iota\omega_{i_\gamma})$ as $\hat{\mathfrak{t}}$-modules. Thus, they must be isomorphic as $\hat{\mathfrak{g}}$-modules. Or equivalently, they are isomorphic as modules over $\{\mathcal{F}_n\}$.

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